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# STAT 131: Quiz 8 *85 total points*

Name: ____________________________________________

Someone offers you the possibility to play a gambling game with the following rules. First, you decide how much money you're willing to put at risk in this game: this amount — let's call it $A > 0$ — is referred to as your *stake* (all the monetary quantities are in dollars in this problem); think of $A$ as a fixed positive real number in what follows. Having chosen your stake, you're allowed to bet any amount $0 \le B \le A$ (thus, as a decision problem, your possible actions in this situation correspond to values of $B$). If you win the bet, which occurs with probability $0 < p < 1$, your stake becomes $(A + B)$; if you lose, it becomes $(A - B)$, and this (of course) occurs with probability $(1 - p)$; and (crucially) *$p$ is known to you.* Let $X$ denote the value of your stake after the gamble has occurred. The point of this problem is to explore your optimal betting strategy, *assuming the Principle of Maximization of Expected Utility (MEU)*, for two different reasonable-looking utility functions, to see which one leads to more sensible behavior.

(a) Write out the probability mass function (PMF) for $X$. *5 points*

(b) First let's suppose that for you, utility coincides with money, i.e., $U_1(x) = x$. Work out your expected utility $E[U_1(X)]$ as a function of $A, B$ and $p$. Holding $A$ and $p$ constant, what type of function of $B$ is this? Explain briefly, and sketch this function for $0 \le B \le A$ in the three separate cases $(p < \frac{1}{2}), (p = \frac{1}{2}), (p > \frac{1}{2})$. *15 points*

(c) Now let's maximize $E[U_1(X)]$ as a function of $B$.

(i) By considering the three different cases for $p$ mentioned above, and with particular reference to your sketches in (b), briefly explain why $B_1^*$, the optimal $B$ under $U_1$, is as follows:

$$B_1^* = \left\{ \begin{array}{cc} 0 \text{ (don't bet)} & \text{for } \ p < \frac{1}{2} \\ \left( \begin{array}{c} \text{bet any number} \\ \text{between 0 and } A \end{array} \right) & p = \frac{1}{2} \\ A \text{ (bet it all)} & p > \frac{1}{2} \end{array} \right\} \qquad (1)$$

*10 points*

(ii) Compute the first partial derivative of $E[U_1(X)]$ as a function of $B$, and try setting it equal to 0 and solving for $B$. With reference to your sketches in (b), briefly explain why this standard calculus approach to maximizing a function won't work in this problem. *10 points*

(iii) Identify one feature of this betting strategy that seems reasonable, and two features that seem unreasonable; explain briefly. *10 points*

(d) Now let's suppose instead that you use Daniel Bernoulli's utility function $U_2(x) = 1 + \log(x)$.

(i) Work out your expected utility $E[U_2(X)]$ as a function of $A, B$ and $p$. Take $A = 10$ for illustration and get `Wolfram Alpha` (or some other equivalent environment) to plot this as a function of $B$ from $-5$ to $A$ for two different values of $p$: 0.4 and 0.7; reproduce these plots in your solutions, either electronically or by hand-sketching what your software environment showed you. Holding $A$ and $p$ constant, what type of function of $B$ is this (increasing, decreasing, concave, convex)? Explain briefly. *10 points*

(ii) Compute the first partial derivative of $E[U_2(X)]$ with respect to $B$, set this expression to 0, and solve for $B$. In many problems this would be the optimal $B^*$ (the value of $B$ that maximizes the expected utility), but the situation is a bit more subtle here. Identify the range of values of $p$ for which the $B^*$ you obtained by differentiation cannot be the solution to this optimality problem, and briefly explain why. *10 points*

(e) By examining the two different cases $(p \leq \frac{1}{2}), (p > \frac{1}{2})$, briefly explain why $B_2^*$, the optimal $B$ under $U_2$, is as follows:

$$B_2^* = \left\{ \begin{array}{cc} \text{0 (don't bet)} & \text{for} \quad p \leq \frac{1}{2} \\ \left( \begin{array}{c} 2\left(p - \frac{1}{2}\right) A \\ \text{(bet an amount} \\ \text{between 0 and } A \\ \text{in proportion to} \\ \text{how much bigger} \\ p \text{ is than } \frac{1}{2}) \end{array} \right) & p > \frac{1}{2} \end{array} \right\}, \qquad (2)$$

Identify two features of this betting strategy that seem eminently reasonable; explain briefly. *10 points*

(f) Is it fair to describe one of the two betting strategies above, based on the two different utility functions, as more risk-seeking than the other one? If so, which is which? If not, why not? Explain briefly. *5 points*