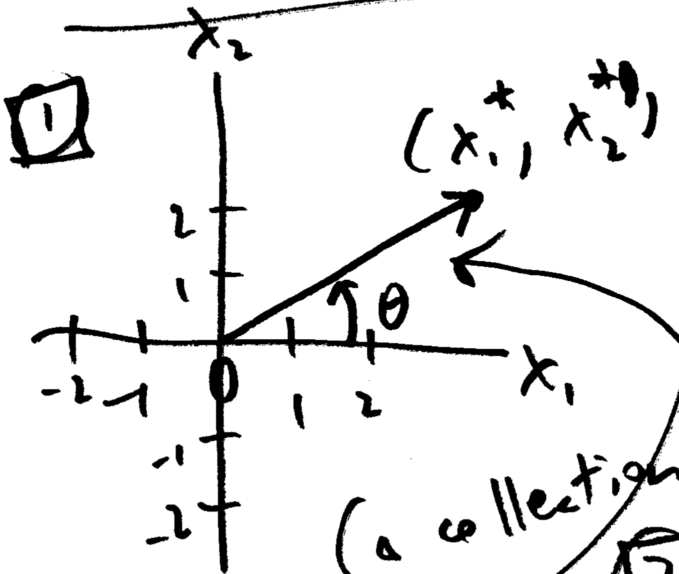
Eigenvalues & eigenvectors of a square matrix: linear algebra (review)



[1] the Euclidean plane ($\mathbb{R}^2$) is an example of a **vector space**. (a collection of vectors + arithmetic rules)

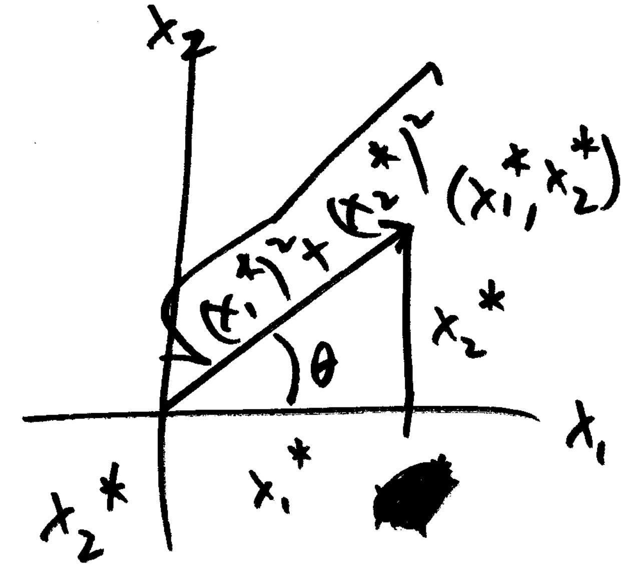
[2] In 2 dimensions a **vector** is just an ordered pair of Cartesian coordinates

$$\underset{2 \times 1}{x^*} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}_{2 \times 1} = \begin{pmatrix} a \\ b \end{pmatrix}$$

\# rows → 2, # columns → 1: 2 rows and 1 column: a column vector

visualize as an arrow from the origin (0, 0) to the point (x_1^*, x_2^*). Thus a vector has magnitude (length) and (orientation) direction

By Pythagoras the vector $\underset{\sim}{x}_1 = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ ②



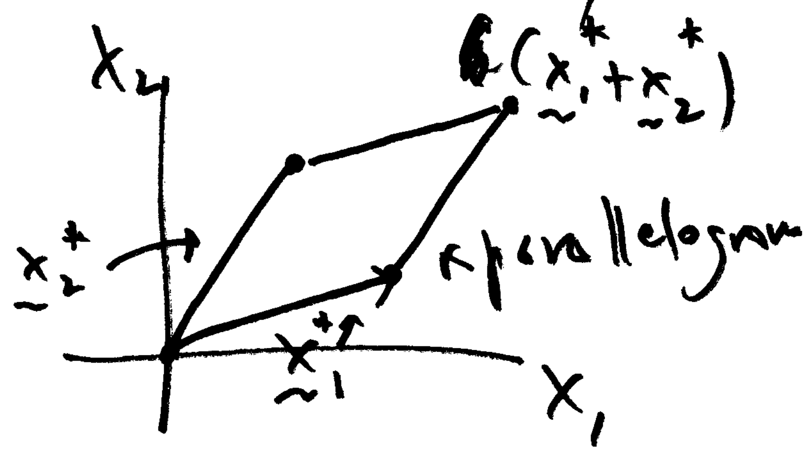
has length $\sqrt{(x_1^*)^2 + (x_2^*)^2}$ (magnitude)

and its angle θ is such that $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{x_2^*}{x_1^*}$

(relative to the x_1 axis)

$\boxed{3}$ arithmetic with vectors

vector arithmetic is motivated by



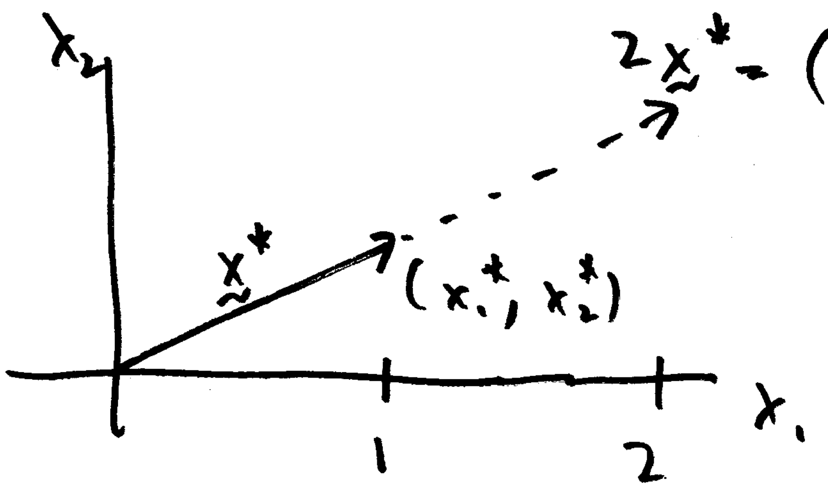
<u>physics</u>: if

$\underset{\sim}{x}_1^* = \begin{pmatrix} x_{11}^* \\ x_{21}^* \end{pmatrix}$ and

$\underset{\sim}{x}_2^* = \begin{pmatrix} x_{12}^* \\ x_{22}^* \end{pmatrix}$ then $\underset{\sim}{x}_1^* + \underset{\sim}{x}_2^* = \begin{pmatrix} x_{11}^* + x_{12}^* \\ x_{21}^* + x_{22}^* \end{pmatrix}$

(just add the components)

A <u>scalar</u> is just a real number c (this could be considered as a 1 by 1 vector $\left(c \right)$, but people don't write a scalar that way usually.



In physics people often think of vectors as representing <u>forces</u> (put a billiard ball at the origin & give it a <u>push</u> away from the origin: it will follow a straight line & eventually stop (friction); $\underset{\sim}{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$ is where it comes to rest)

So the natural way to define the operation of multiplying a vector by a scalar is just $c\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} c x_1^* \\ c x_2^* \end{pmatrix}$.

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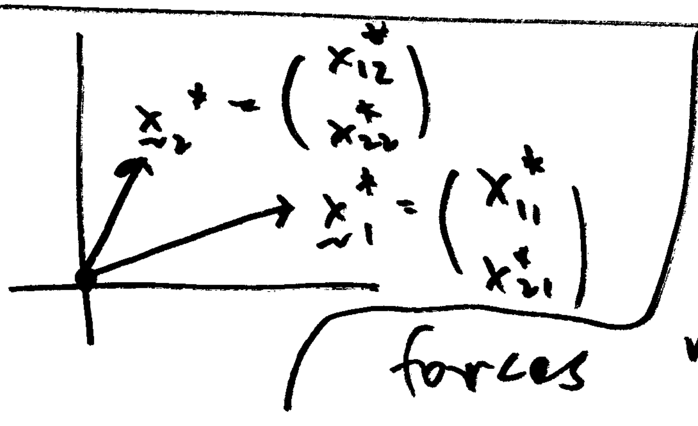
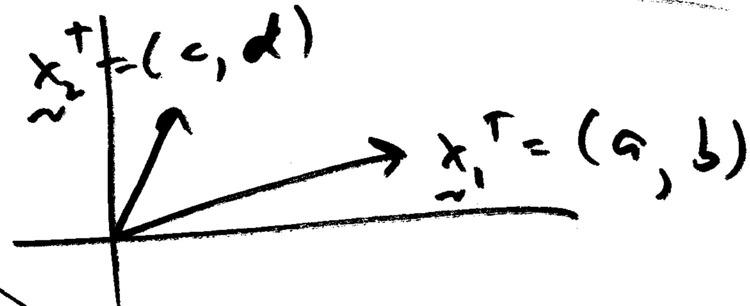
multiplying two vectors This is the least intuitive of the arithmetic operations on vectors, but it has a good story. There are several ways to meaningfully define the product of 2 vectors: the most useful are (dot product) and (cross product) (we won't look at this)

Dot product



How combine multiplicatively?

<u>definition</u>: the <u>transpose</u> of a <u>column</u> vector $\underset{2}{\underset{\sim}{x}_1} = \underset{1}{\begin{pmatrix} a \\ b \end{pmatrix}}$ is the vector $\underset{1}{\Big(}\underset{\sim}{x}_1^{T}\underset{2}{\Big)} = (a, b)$ formed by making it a <u>row</u> vector (laying it on its side)

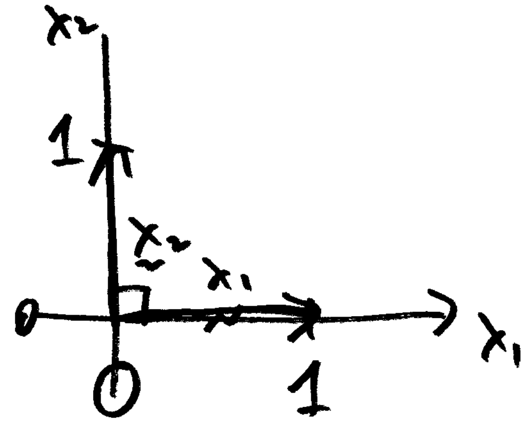


Physics question:

Simultaneously shove the billiard ball with both $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$; how much extra force will result in the $\underset{\sim}{x}_1$ (~~[illegible]~~) direction? $\underset{\sim}{x}_2$?

Define this to be the <u>dot</u> product of $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$

Hard to see the general answer.



Special case: $\underset{\sim}{x}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\underset{\sim}{x}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ (these are called <u>unit vectors</u> (length 1) in the directions of the coordinate axes.

$\boxed{\text{Q:}}$ In this case, how much extra force in the $\underset{\sim}{x}_1$ direction will result from also shoving in the $\underset{\sim}{x}_2$ direction (and vice versa)?

$\boxed{\text{A:}}$ 0, because $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are <u>perpendicular</u>

Note that the answer is a scalar, i.e., we want the dot product of ${}_{2}\underset{\sim}{x}_{1}$ and ${}_{2}\underset{\sim}{x}_{2}{}_{1}$ to be 1×1.

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Q: How combine $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x_2$ in a way that (a) involves multiplication and (b) yields 0?

~~scalar~~ multiplication

$$\begin{array}{ccccl} (1) & \times & (0) & = & 0 \; + \\ (0) & \times & (1) & = & 0 \\ \hline & & & & 0 \end{array}$$

Conjecture: Another way to write this: transpose x_1 to get $x_1^T = (1, 0)$;

now define $\underset{1 \times 2}{x_1^T} \cdot \underset{2 \times 1}{x_2} = \underset{1 \times 2}{(1, 0)} \cdot \underset{2 \times 1}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$= 1 \cdot 0 + 0 \cdot 1 = 0$

Note (check) that

$x_1^T \cdot x_2 = x_2^T \cdot x_1$ so that the conjectured definition is symmetric in its inputs (important).

This conjecture turns out to be highly useful

Definition (dot product of two vectors)

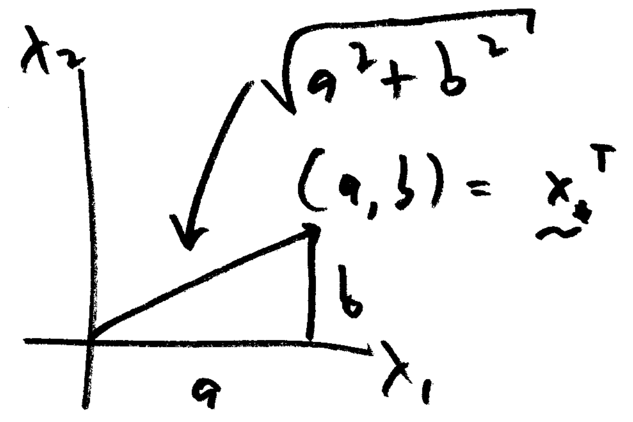
$$\underset{2\ \ 1}{\underset{\sim}{x_1}} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \underset{2\ \ 1}{\underset{\sim}{x_2}} = \begin{pmatrix} c \\ d \end{pmatrix} \qquad \text{Then}$$

$$\boxed{\underset{1\ \ 2}{\underset{\sim}{x_1}^T} \cdot \underset{2\ \ 1}{\underset{\sim}{x_2}} \qquad (a, b) \cdot \begin{pmatrix} c \\ d \end{pmatrix}} = \boxed{\underset{\sim}{x_2}^T \cdot \underset{\sim}{x_1} \qquad (c, d) \begin{pmatrix} a \\ b \end{pmatrix}} = a \cdot c + b \cdot d$$

scalar multiplication

scalar addition

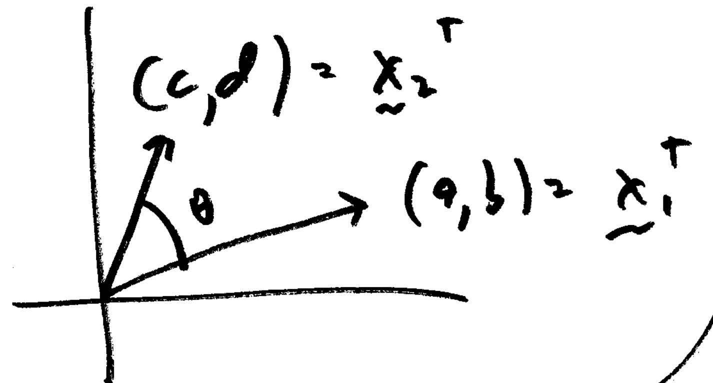
Definition The (Euclidean) <u>norm</u> of a vector $\underset{\sim}{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ is



norm of $\underset{\sim}{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

is $\|\underset{\sim}{x}\| = \sqrt{a^2+b^2}$

(its length)



can show (this connection is obscure for now)

that another way to compute the dot product is $\underset{\sim}{x}_1^T \underset{\sim}{x}_2 = \|\underset{\sim}{x}_1\| \cdot \|\underset{\sim}{x}_2\| \cdot \cos\theta$

Matrices These are just column vectors stacked together from left to right:

$$\underset{2\ \ 2}{A} = \left(\begin{array}{c|c} a & c \\ b & d \end{array}\right)$$

$$\begin{array}{cc} \underset{\sim}{x}_1 , & \underset{\sim}{x}_2 \end{array}$$

(or row vectors stacked from top to bottom)

~~(a rectangular table, like a spreadsheet)~~

A general matrix ${}_{r}A_{c}$ with r rows and c columns can be thought of in two ways: c column vectors, each (r by 1), or r row vectors, each (1 by c)

simple results with

For Markov chains we only need to think about square matrices, for which $(r = c)$

Matrix arithmetic (ex. $r = c = 2$)

Adding two matrices

$$({}_{2}A_{1})_{2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \quad ({}_{2}A_{2})_{2} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} ;$$

use the definition of vector addition to get

$$ {}_{2}(A_1 + A_2)_{2} = \left(\begin{array}{c|c} a+e & b+f \\ \hline c+g & d+h \end{array}\right)$$

I should have mentioned quite a while ago: it does not make sense to add or multiply vectors or matrices that do not conform:

Def. Two column vectors $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ conform if they have the same number of rows;

$$\underset{\sim}{x}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \underset{\sim}{x}_2 = \begin{pmatrix} c \\ d \\ e \end{pmatrix}$$

$${}_{2}(\underset{\sim}{x}_1)_{1} + {}_{3}(\underset{\sim}{x}_2)_{1} \quad \text{and} \quad {}_{1}(\underset{\sim}{x}_1^T)_{2} \cdot {}_{3}(\underset{\sim}{x}_2)_{1}$$ don't make sense

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Similarly (of course) for matrices:

$${}_{2}(A_1)_{2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad {}_{2}(A_2)_{3} = \begin{pmatrix} e & f & g \\ h & i & j \end{pmatrix}$$

A_1 & A_2 don't conform (can't add or multiply)