This catch-up time: topics next time:

Discrete-time Markov chain with finite state space $S = \{1, 2, \ldots, k\}$

Initial distribution at time 1

$v = (v_1, v_2, \ldots, v_k)$

$s_0 = v_0 = \frac{1}{k} \sum v_i = 1$

$p = \text{one-step transition matrix}$

$k \times k$

$(i,j)$ entry of $p$

$p_{ij} = P(X_{n+1} = j \mid X_n = i) \quad \text{(constant)}$

(time homogeneous)

$p(X_1 = i) = v_i, \quad i = 1, \ldots, k$

$p(X_2 = j) = ?$ hard to think about

Marginal, so get help:

$X_2$ depends on $X_1$, so partition over $X_1$ and use the LP $p$: 
\[ P(\xi_2 = j) = \frac{1}{k} \sum_{i=1}^{k} \mathbb{P}(\xi_2 = j), \xi_1 = i) \]

\[\begin{array}{c}
\begin{array}{c}
\xi_1 = 1 \\
\xi_1 = 2 \\
\xi_1 = 3 \\
\xi_2 = 1 \\
\xi_2 = 2 \\
\xi_2 = 3 \\
\xi_2 = 4 \\
\end{array}
\end{array}\]

\[ = \frac{k}{2} \sum_{i=1}^{k} P(\xi_1 = i) \beta(\xi_2 = j | \xi_1 = i) \]

\[ = \frac{k}{2} \sum_{i=1}^{k} v \cdot p_{ij} \]

(looks like vector dot product)

\text{Take } j = 1 \]

\[ P(\xi_2 = 1) = \frac{1}{k} \sum_{i=1}^{k} v_i \cdot p_{ij} = \]

\[ (v_1, v_2, \ldots, v_k) \begin{pmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{k1} \end{pmatrix} \]

This works for all \( j = 1, \ldots, k \).

Collect results together in matrix notation:
\[(\nu_1, \ldots, \nu_k) \cdot \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix} \sim \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_k \end{pmatrix} = \nu \mathbf{p}, \quad \sim_k = 1\]

Distribution of where the chain is at time 2 is \(\nu\); at time 2 it's \(\nu \mathbf{p}\); by exactly the same reasoning at time 3 it's \(\nu \mathbf{p}^2\); \(\ldots\); at time \(m\) it's \(\nu \mathbf{p}^{m-1}\).

By question: does \(\lim_{m \to \infty} \nu \mathbf{p}^{m-1}\) exist? Exso it would be the equilibrium distribution.
So 2 ways to find equilibrium distribution:

1. See if \( \lim_{n \to \infty} P^m \) exists.

2. Find \( \mathbf{v} \) such that \( \mathbf{v} P = \mathbf{v} \), i.e., find a left eigenvector of \( P \) with (left) eigenvalue \( 1 \) \( \iff \) find a right eigenvector of \( P^T \) (transpose) with right eigenvalue \( 1 \).

- Suppose \( \mathbf{y} = c \mathbf{x} \) \( c \neq 0 \):
  \[
  A \mathbf{y} = A(c \mathbf{x}) = c A \mathbf{x} = c \lambda \mathbf{x} = 0 \mathbf{x},
  \]
  \[
  \implies A \mathbf{x} = \mathbf{0}.
  \]
$p$-step transition matrix $k = 1$

\[ \sum_{k=1}^{p} \lambda_k \]

the $\lambda$ that come out of your eigenanalysis software will have $0 \leq \lambda_i$

but not necessarily

\[ \sum_{k=1}^{p} \lambda_k = 1 \]

let $\lambda_i$ with

just divide $\lambda_k$

\[ \sum_{k=1}^{p} \lambda_k \]
Adult human height measurements like (conceptually) continuously on \( \mathbb{R}^+ \) bivariate PDF

\[
f_{x_1, x_2}(x_1, x_2) = \frac{1}{{\sigma_1} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 \right]
\]

Level sets (contours): \( f_{x_1, x_2}(x_1, x_2) = c \)
\( x_2 \) = \( \beta_0 + \beta_1 x_1 \)

\( \bar{x}_2 - \beta \bar{x}_1 \)

\( x_2 = \bar{x}_2 + r \frac{s_2}{s_1} (x_1 - \bar{x}_1) \)

\( \mathbb{E} \left( \frac{\bar{x}_2}{\bar{x}_1} \right) \sim \text{univariate normal} \)

\( V \left( \frac{\bar{x}_2}{\bar{x}_1} = \chi_i^* \right) \)

\( = \sigma_2^2 (1 - \rho^2) \)

\( r \)

\( \beta \)

\( \sigma \)

\( s \)

\( \text{GRE} \)

\( \text{score} \)

\( \text{time} \)

\( \text{1st time} \)