this catch-up time: topics next time: ↓

Discrete time Markov chain with finite state space $S = \{1, 2, \ldots, k\}$ ← (labels)

STAT 131
4 Jun 20
(lecture)

①

$\underset{1\times k}{\underset{\sim}{v}} = (v_1, v_2, \ldots, v_k)$ initial distribution at time 1

$\left(0 \le v_i \le 1, \ \sum_{i=1}^{k} v_i = 1\right)$

$\underset{k\times k}{P}$ = one-step transition matrix:

(i,j) entry of P

$p_{ij} = P(X_{n+1} = j \mid X_n = i)$ (constant in n)

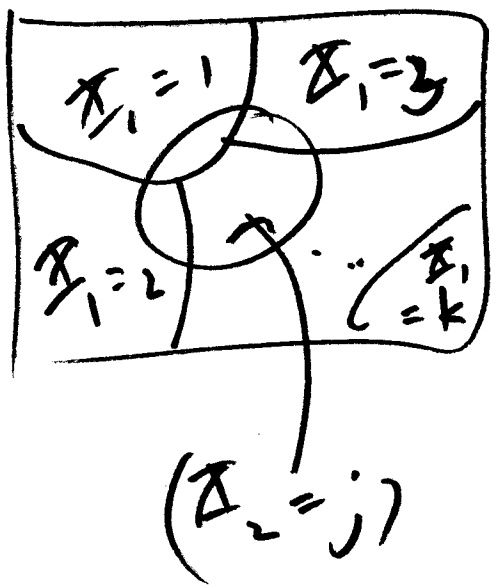
(time homogeneous)

$P(X_1 = i) = v_i, \quad i = 1, \ldots, k$

$\boxed{P(X_2 = j) = ?}$ ← hard to think about marginally, so get help:

X_2 depends on X_1, so partition over X_1 and use the LTP:

$$P(X_2 = j) = \sum_{i=1}^{k} P(X_2 = j, X_1 = i)$$



$$= \sum_{i=1}^{k} P(X_1 = i)\, P(X_2 = j \mid X_1 = i)$$

$$= \underbrace{\sum_{i=1}^{k} v_i \, p_{ij}}_{\text{looks like vector dot product}}$$

Take $j = 1$ $\quad P(X_2 = 1) = \sum_{i=1}^{k} v_i \, p_{i1} =$

$$(v_1, v_2, \ldots, v_k) \begin{pmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{k1} \end{pmatrix}$$

1st column of $\underline{\underline{P}}$

This works for all $j = 1, \ldots, k$; collect results together in matrix notation:

$$(v_1, \dots, v_k) \cdot \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix} = \underset{\sim}{v}\,\underset{=}{P}$$

5a) Distribution of where the chain is at time 1 is $\underset{\sim}{v}$; at time 2 it's $\underset{\sim}{v}\,\underset{=}{P}$; by exactly the same reasoning at time 3 it's $\underset{\sim}{v}\,\underset{=}{P}^2$; ...; at time m it's $\underset{\sim}{v}\,\underset{=}{P}^{m-1}$

big question: does $\lim_{m \to \infty} \underset{\sim}{v}\,\underset{=}{P}^{m-1}$ exist? If so it would be the equilibrium distribution

So 2 ways to find equilibrium distribution: ① see if $\lim_{m\to\infty} \underset{\sim}{v}_{k=1} P^{m-1}$ exists

② find $\underset{1\sim k}{v}$ such that $\underset{1\sim k}{v}\, \underline{\underline{P}} = \underset{1\sim k}{v}$, i.e. find a left eigenvector of $\underline{\underline{P}}$ with (left) eigenvalue 1 $\longleftrightarrow$ find a right eigenvector of $\underline{\underline{P}}^T$ (transpose $\underline{\underline{P}}$) with right eigenvalue 1

Suppose can find $\underset{k\sim 1}{x}$ such that $\underset{k}{A}\, \underset{k\sim 1}{x} = \lambda \underset{k\sim 1}{x}$

$z = c\underset{\sim}{x}$, $c \neq 0$

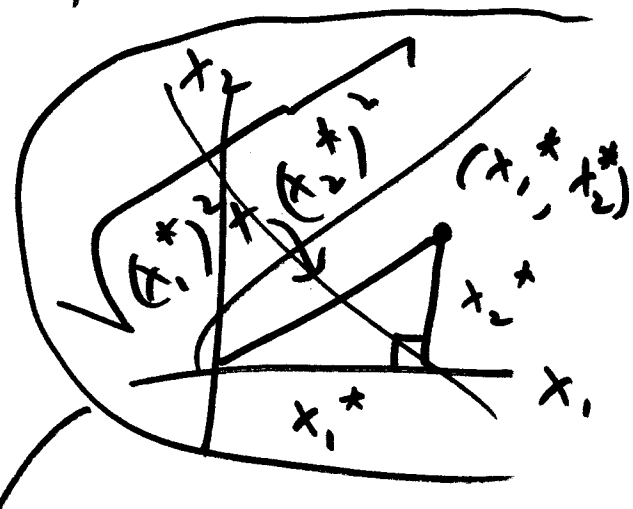
$$A z = A(c\underset{\sim}{x}) = c A \underset{\sim}{x} = c\lambda \underset{\sim}{x} = \lambda (c\underset{\sim}{x})$$

$$= \lambda z$$

$\underset{k=k}{P}$ 1-step transition matrix

$$\underset{1}{\underset{\sim}{v}}\ \underset{k}{\underset{=}{P}} = 1 \cdot \underset{\sim}{v}$$

the $\underset{\sim}{v}$ that comes out of your eigenanalysis software will have $0 \le v_i$



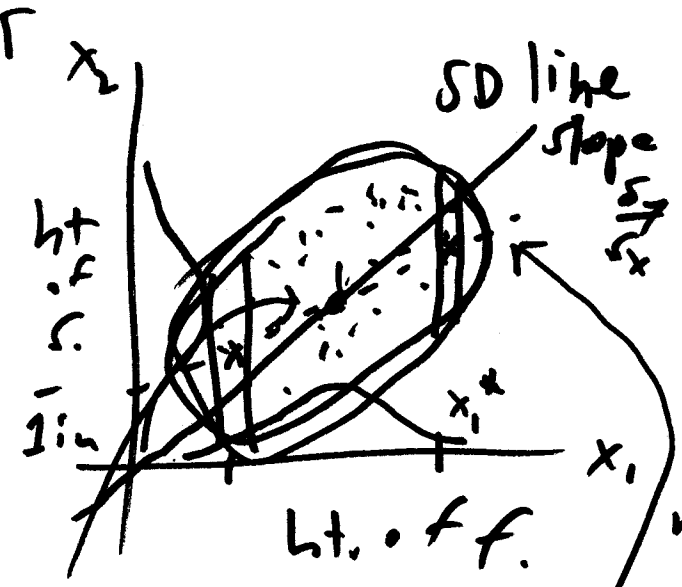
but not necessarily

$$\sum_{i=1}^{k} v_i = 1 \text{ ; to}$$

$$\|\underset{1}{\underset{\sim}{x}}_k\| = \sqrt{\sum_{i=1}^{k} x_i^2}$$

set a $\underset{\sim}{v}^*$ with $\sum_{i=1}^{k} v_i^* = 1$,

just divide $\dfrac{\underset{\sim}{v}}{\sum_{i=1}^{k} v_i}$



$(\bar{x}_1, \bar{x}_2)$

ht. of son in x_2 | ht. of f. x_1

$r \doteq 0.65$

$n \doteq 1050$

mean $\bar{x}_2 = 68.4$ in, $\bar{x}_1 = 68.4$ in

SD $s_{x_2} = 2.5$ in, $s_{x_1} = 2.5$ in

regression line for predicting x_2 from x_1

prediction estimate

$$\hat{x}_2 = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

$$\hat{\beta}_1 = r \cdot \frac{s_y}{s_x}$$

$$\hat{\beta}_0 = \bar{x}_2 - \hat{\beta}_1 \bar{x}_1$$

Two (B.) variate normal distribution

eugenics

pop. all British families in 1890 similar to those in Galton's sample in all relevant ways

sample

(X_1, X_2) r.v.

x_{12} x_{11}

like DLD

$\bar{x}_2$ $\bar{x}_1$

s_{x_2} s_{x_1}

r

N

pop. corr. ρ

mean μ_{x_2} μ_{x_1}

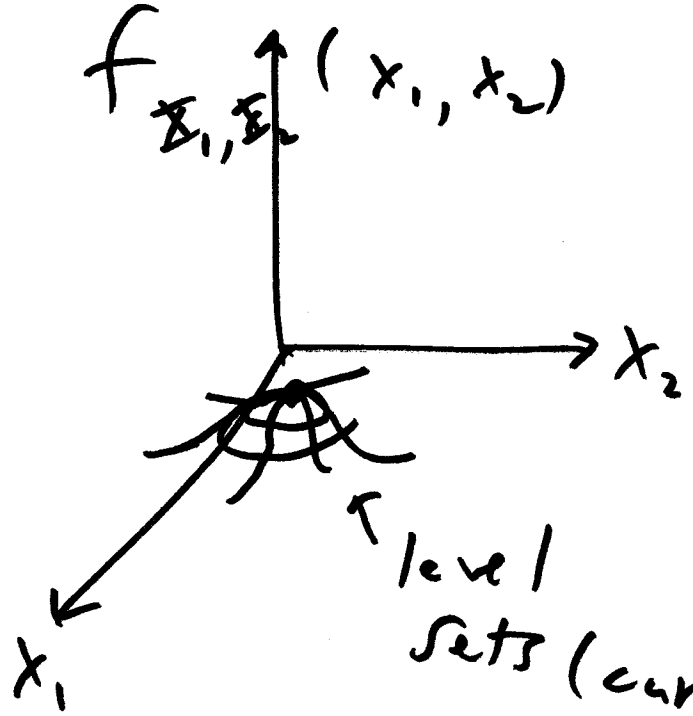
SD σ_{x_2} σ_{x_1}

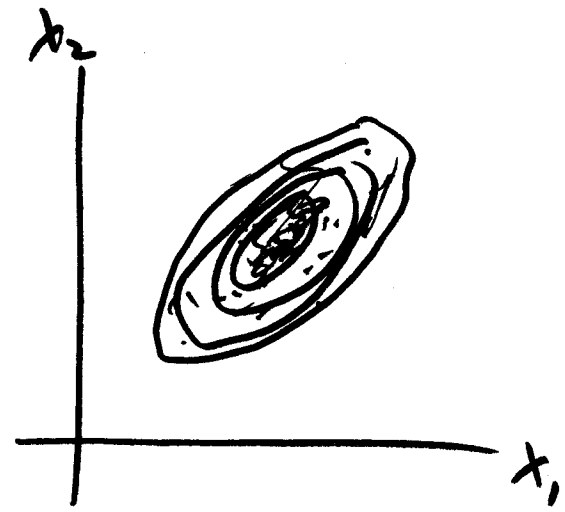
adult human height measurements live (conceptually) continuously on $\mathbb{R}^+$

~~bivariate PDF~~

$$f_{X_1, X_2}(x_1, x_2) = ?$$

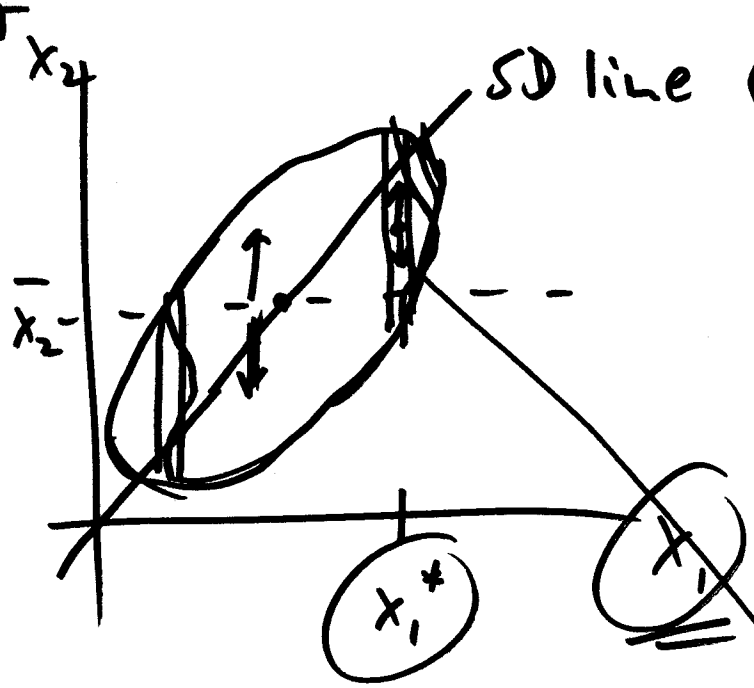
$$f_{X_1}(x_1) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2\right]$$





set $f_{X_1, X_2}(x_1, x_2) = c$

& solve for (x_1, x_2)



$(X_2 \mid X_1) \sim$ univariate normal

$$V(X_2 \mid X_1 = x_1^*) = \sigma_2^2 \cdot (1 - \rho^2)$$

$$\hat{x}_2 = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

$$\hat{\beta}_1 = r \frac{s_2}{s_1}$$

$$\bar{x}_2 - \hat{\beta}_1 \bar{x}_1$$

$$= \bar{x}_2 + r \frac{s_2}{s_1}(x_1 - \bar{x}_1)$$