Discussion section, week of 4–8 May 20

**Practice with bivariate PDFs: joint, marginal and conditional densities**

STAT 131
4 May 20

① Consider joint PDF

$$f_{X,Y}(x,y) = \begin{cases} A\,x^2 y^2 & \text{for } 0 \le x^2 + y^2 \le B \\ 0 & \text{else} \end{cases} \quad (*)$$

$$A = \frac{24}{\pi B^3}$$

support $S_{X,Y}$

① Could this be a bivariate PDF with appropriate choices for $A$, $B$?

3 criteria for a function $f_{X,Y}(x,y)$ to be a bivariate PDF:

✓ ⓪ cont.

✓ ① $f_{X,Y}(x,y) \ge 0$

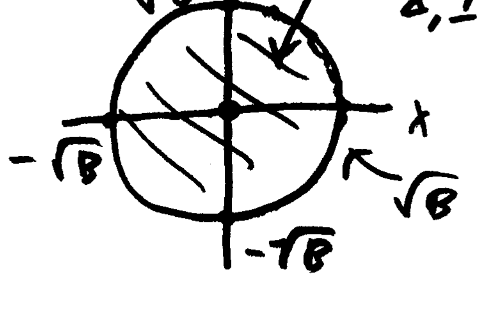
✓ ② $\displaystyle\iint_{S_{X,Y}} f_{X,Y}(x,y) = 1$

so $A$, $B$ must satisfy:

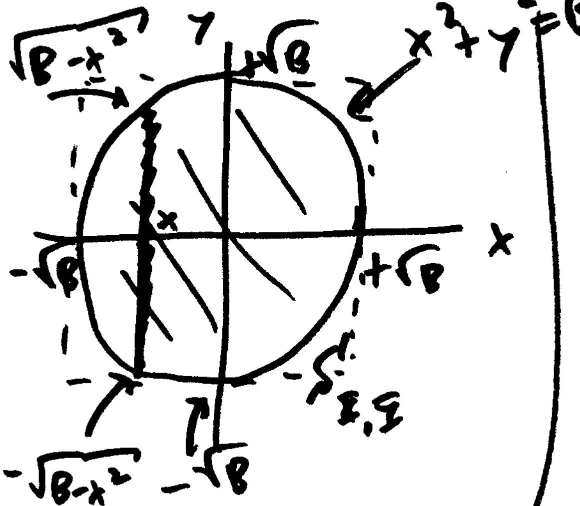
$A \ge 0$, $B \ge 0$ and

$$\iint_{S_{X,Y}} A x^2 y^2 \, dy\, dx = 1$$

② Sketch the bivariate support set $S_{X,Y} = \{(x,y) : 0 \le x^2 + y^2 \le B\}$




all points in $\mathbb{R}^2$ on or inside circle




② Identify the marginal support sets $S'_X$ and $S'_Y$:

$S'_X = [-\sqrt{B}, +\sqrt{B}]$

$S'_Y = [-\sqrt{B}, \sqrt{B}]$ $\quad 0 \le x^2+y^2 \le B$

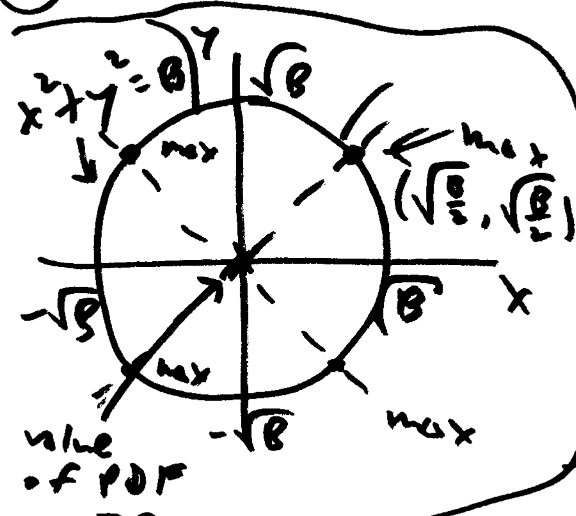
$(x^2+y^2=B) \to y = \pm\sqrt{B-x^2}$

③ Compute the normalizing constant:

$$1 = \iint_{S'_{X,Y}} A x^2 y^2 \, \underline{\underline{dy\, dx}} = \int_{-\sqrt{B}}^{\sqrt{B}} \left[ \int_{-\sqrt{B-x^2}}^{\sqrt{B-x^2}} A x^2 y^2 \, dy \right] dx$$

$\boxed{\text{Wol}}$ $=$ $1 = \frac{\pi A B^3}{24}$ $\quad$ So $A = \frac{24}{\pi B^3}$

④ Visualize the contour plot of $f_{X,Y}$:




where is $x^2y^2$ smallest on or inside the circle?

0

where is $x^2y^2$ largest on or inside the circle?

$g(x) = x^2(B-x^2)$

maximized at $x = \pm\sqrt{\frac{B}{2}}$

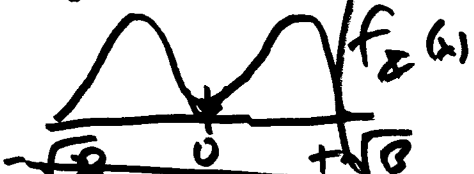
⑤ Compute the marginal PDF for $X$:

for $-\sqrt{B} \le x \le +\sqrt{B}$ $\quad f_X(x) = \int f_{X,Y}(x,y)\,dy$

$$= \int_{-\sqrt{B-x^2}}^{\sqrt{B-x^2}} \frac{24}{\pi B^3} x^2 y^2 \, dy$$

So

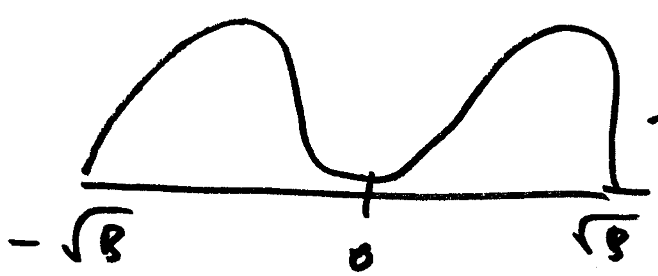
$$f_X(x) = \begin{cases} \dfrac{16x^2(B-x^2)^{3/2}}{\pi B^3} & \text{for } -\sqrt{B} \le x \le \sqrt{B} = S_X \\ 0 & \text{else} \end{cases}$$




sketch & check that it integrates to 1

⑥ Compute or identify the marginal PDF for $Y$: by symmetry

for $-\sqrt{B} \le y \le \sqrt{B}$ $\quad f_Y(y) = \begin{cases} \dfrac{16y^2(B-y^2)^{3/2}}{\pi B^3} & \text{for } y \in S_Y \\ 0 & \text{else} \end{cases}$



(7) Are $X$ and $Y$ independent or dependent in this joint distribution?

$$f_{X,Y}(x,y) = \begin{cases} \frac{24}{\pi B^3} x^2 y^2 & \text{for } 0 \le x^2 + y^2 \le B \\ 0 & \text{else} \end{cases}$$

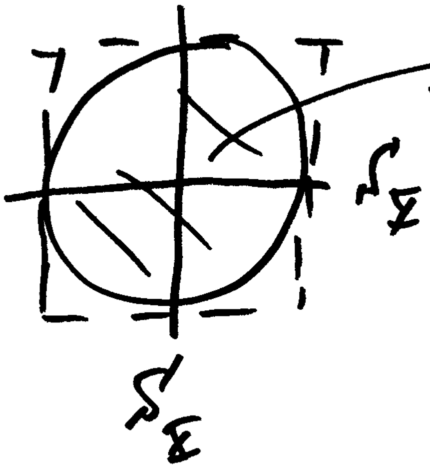
$X, Y$ indep iff $f_{X,Y}(x,y) \stackrel{S_{X,Y}}{=} f_X(x) \cdot f_Y(y)$

for T/F statements $A, B$: $A, B$ indep. iff $P(A \text{ and } B) = P(A) \cdot P(B)$

A reminder of the definition of conditional probability for events / true-false propositions:

$$P(B \mid A) = \begin{cases} \frac{P(A \text{ and } B)}{P(A)} & \text{if } P(A) > 0 \\ \text{undefined} & P(A) = 0 \end{cases}$$

It turns out (see lecture soon) that there's a completely analogous result for random variables:




$\{0 \le x^2 + y^2 \le B\} = S_{X,Y}$

$X, Y$ dependent in this joint dist.

The conditional PDF of $Y$ given $X$ is: for a fixed $x$ with $f_X(x) > 0$

$$f_{Y|X}(y|x) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{f_X(x)} \\ \dfrac{3}{2}\dfrac{y^2}{(B-x^2)^{3/2}} \quad \text{for } -\sqrt{B-x^2} \le y \le +\sqrt{B-x^2} \end{cases}$$

here

for $-\sqrt{B} \le x \le +\sqrt{B}$

$$f_{Y|X}(y|x) = \frac{3\,\dfrac{\cancel{24}}{\cancel{\pi B^3}}\,\cancel{x^2}\,y^2}{\dfrac{2\,\cancel{16}\,\cancel{x^2}\,(B-x^2)^{3/2}}{\cancel{\pi B^3}}}$$