Discussion section week of 20 - 24 Apr 20

Case Study: Monty Hall

(on course webpage)

STAT 131 20 Apr 20

(1) structure algorithm for of all applied math problem-solving (meta-code)

real world	math world
start here	
step 1	step 2
state problem $\mathbb{P}$ clearly and accurately in (natural language)	translate → express $\mathbb{P}$ in math language completely and accurately
	step 3 solve
real-world solution	obtain math solution in math language
finish here	step 4 translate ←
express math solution in real-world terms (natural language)	

Example of how inaccurate translation (2)
from natural language to math language
leads to a dead end ~~in a probability calculation~~ Take-home test 1 problem 3

Crucial first step

(1) Identify how many different sources of information there are in the (natural-language) problem statement, and create symbols to stand for all relevant (true/false) propositions.

Here there are 2 different kinds of information:

$\boxed{1}$ who will be pardoned

$\boxed{2}$ who the warden says (tells A) will NOT be pardoned

If we let
A = (A will be pardoned) and similarly for B and C, you might think that the relevant (desired) probability, after A has heard

what the warden said, is $P(A \mid \text{not } B)$ but this is not right: A has not learned that B will not be pardoned; all A has learned is that (warden ~~says~~ B will not be pardoned)

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we need new symbols for information of type $\boxed{2}$ above. Let ($W = w$) stand for (Warden says (tells A) w will not be pardoned)

warden (capital w)

Then what we want is $P(A \mid W = B)$, and this is not at all the same thing as $P(A \mid \text{not } B)$. Monty Hall case study

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In full generality there are three

different kinds of information here:

$\boxed{1}$ which door you initially choose; $\mathcal{I}$

$\boxed{2}$ which door Monty opens to show you a goat; and $\boxed{3}$ where the car really is. C M

So let's let $\mathcal{I}_i = \left(\text{you chose door } i\right)$ $(i = 1, 2, 3)$

$M_j = \left(\begin{array}{c}\text{Monty reveals}\\ \text{goat behind}\\ \text{door } j\end{array}\right)$ $(j = 1, 2, 3)$

$C_k = \left(\begin{array}{c}\text{car really is}\\ \text{behind door } k\end{array}\right)$ $(k = 1, 2, 3)$

WLOG

Without loss of generality let's ~~[illegible]~~ look (condition on) at $\mathcal{I}_1$ and M_2;

why? symmetry/invariance

we want to compute

$P(C_3 \mid \mathcal{I}_1 \text{ and } M_2)$ and compare it with

$P(C_1 \mid \mathcal{I}_1, M_2)$. Naive intuitive

argument: since Monty has now shown

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you a goat behind door 2, it's become a 50/50 proposition between C_1 and C_3, so there's no advantage in switching:

$$P_{\substack{\text{intuitive}\\ \text{but wrong}}}(C_3 \mid \mathcal{I}_1, M_2) = \frac{1}{2}$$

warning: probability is tricky.

$$= P_{\substack{\text{intuitive}\\ \text{but wrong}}}(C_1 \mid \mathcal{I}_1, M_2)$$

The above argument is guesswork, not math; let's try math. $\boxed{\text{step I}}$ given $\mathcal{I}_1$ and M_2, under the rules of the game the only possibilities left for C_k are C_1 and C_3 (why?), so C_1 and C_3 have become opposites:

C_2: Monty shows you car

$$P(C_1 \mid \mathcal{I}_1, M_2) + P(C_3 \mid \mathcal{I}_1, M_2) = 1$$

$\boxed{\text{step II}}$ $\mathcal{I}_i$ adds no information to the game; why? $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (but Monty needs to know $\mathcal{I}_2$) ⑥

notice that C_k acts in this problem like the unknown thing (the truth) you're wondering about and M_j acts like data, so $P(C_3 \mid \mathcal{I}_1, M_2)$ is of the form $P\left(\text{unknown} \mid \begin{pmatrix}\text{irrelevant}\\\text{randomization}\end{pmatrix}, \text{data}\right)$

Step III | Notice further that the rules of the game are giving us information of the form $P\left(\underline{\underline{\text{data}}} \mid \begin{pmatrix}\text{irrelevant}\\\text{randomization}\end{pmatrix}, \text{unknown}\right)$

Translating into math, we know these probabilities

$P(M_2 \mid \mathcal{I}_1, C_3) = 1$ | $P(M_2 \mid \mathcal{I}_1, C_1) = \frac{1}{2}$

$P(M_2 \mid \mathcal{I}_1, C_2) = 0$

Step IV | Going from what we know,

$p(m_j \mid I_i, C_k)$,
data, unknown

to what we want,

$p(C_k \mid I_i, m_j)$,
unknown, data

is a job for Bayes's Theorem, and since (from step I) C_1 and C_2 are opposites given (I_1, m_2), let's use Bayes's Theorem in odds ratio form! So far in class the way you've seen this is

$$\underbrace{\left[\frac{p(A \mid B)}{p(\text{not } A \mid B)}\right]}_{\text{posterior odds ratio}} = \underbrace{\left[\frac{p(A)}{p(\text{not } A)}\right]}_{\text{(prior odds ratio)}} \underbrace{\left[\frac{p(B \mid A)}{p(B \mid \text{not } A)}\right]}_{\text{(Bayes factor)}}$$

It's straightforward (if a bit tedious ☹) to show that this extends to situations where there's a third information source (E, say) in the form of a true/false statement that we know to be true: just stick E everywhere to the right of the conditioning bar:

$$\underbrace{\left[\frac{P(A \mid E, B)}{P(\text{not } A \mid E, B)}\right]}_{\text{(posterior odds)}} = \underbrace{\left(\frac{P(A \mid E)}{P(\text{not } A \mid E)}\right)}_{\text{(prior odds)}} \underbrace{\left[\frac{P(B \mid E, A)}{P(B \mid E, \text{not } A)}\right]}_{\text{(Bayes factor)}}$$

In this case study this becomes

$$\left[\frac{P(C_3 \mid \mathcal{I}_1, M_2)}{P(C_1 \mid \mathcal{I}_1, M_2)}\right] = \left[\frac{P(C_3 \mid \mathcal{I}_1)}{P(C_1 \mid \mathcal{I}_1)}\right] \left[\frac{P(M_2 \mid \mathcal{I}_1, C_3)}{P(M_2 \mid \mathcal{I}_1, C_1)}\right]$$

Step V Next, we know that (where the car really is) (C_k) and (which door you initially pick) (I_i) are <u>independent</u>, so the prior odds ratio is ⑨

$$\frac{P(C_3 \mid I_1)}{P(C_1 \mid I_1)} = \frac{P(C_3)}{P(C_1)} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

Step VI But from step III we already have that the (Bayes factor in favor of C_3 over C_1)

$$= \frac{P(M_2 \mid I_1, C_3)}{P(M_2 \mid I_1, C_1)}$$

$$= \frac{1}{\frac{1}{2}} = 2 .$$

So we're done: the posterior odds ratio in favor of C_3 over C_1 is

$$\frac{p(C_3 \mid F_1, m_2)}{p(C_1 \mid F_1, m_2)} = \left(\frac{\frac{1}{3}}{\frac{1}{3}}\right) \cdot \left(\frac{2}{1}\right) = 2 \triangleq o_{C_3}$$

(is defined to be)

and you'll recall that the general way to get from odds ratios o_A to probabilities p_A is

$$p_A = \frac{o_A}{1 + o_A}$$

So finally $p_{C_3} = \dfrac{o_{C_3}}{1 + o_{C_3}} = \dfrac{2}{3}$

and since in this situation $p_{C_3} = p(C_3 \mid F_1, m_2) = 2/3$, you can double your chance of getting the car by switching

Simulation demo

mathwarehouse.com/monty-hall-simulation-online

What was wrong with the previous intuitive argument?

At first glance it seems that Monty showing you a goat behind door 2 (after you initially chose door 1) doesn't give you new information about where the car is, but that's wrong, as the following correct intuitive argument shows:

On those occasions (which happen $\frac{1}{3}$ of the time) on which you (at random, and not known to you) pick the door where the car really is (WLOG let's say door 1) 1, Monty is forced to randomize between (showing you a goat behind door 2) and (showing you a goat behind door 3), i.e., on these occasions you really do learn nothing about the true location of the car);

But on those occasions on which you (at random, and not known to you) don't pick the door where the car really is (and this happens $\frac{2}{3}$ of the time), by showing you a goat behind (say) door 2 he is secretly telling you that the car is behind door 3 (!)

So $\frac{1}{3}$ of the time it doesn't matter if you switch or not, and $\frac{2}{3}$ of the time you get the car by switching; so you should switch. Here's a math argument that formalizes this second intuitive story above. A direct application of Bayes's Theorem in the extended form (for T/F statements)

$$\boxed{P(A \mid E, B) = \frac{P(A \mid E)\, P(B \mid E, A)}{P(B \mid E)}}$$

in this problem is

$$P(C_3 \mid \mathcal{I}_1, n_2) = \frac{P(C_3 \mid \mathcal{I}_1)\, P(n_2 \mid \mathcal{I}_1, C_3)}{P(n_2 \mid \mathcal{I}_1)}$$

Now, as before, $P(C_3 \mid \mathcal{I}_1) = P(C_3) = \frac{1}{3}$

and $P(M_2 \mid \mathcal{I}_1, C_3) = 1$, so

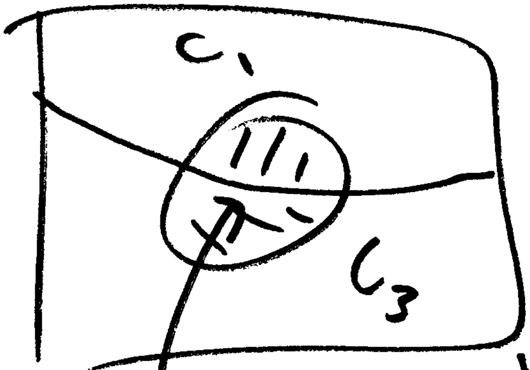
$$\boxed{P(C_3 \mid \mathcal{I}_1, M_2) = \frac{\frac{1}{3} \cdot 1}{P(M_2 \mid \mathcal{I}_1)}}$$

and (as usual) we're stuck with evaluating the annoying denominator/normalizing constant.

(~~meta-code~~)

Crucial observation / data: we can't predict what Monty will do after we've chosen door 1, without knowing the truth about the unknown location of the car (C_k).

Following Dennis Lindley, let's extend the conversation by bringing C_k (truth about unknown) into the calculation by partitioning over it



$(M_2 | I_1)$

$$P(M_2 | I_1) = P(M_2, C_1 | I_1) + P(M_2, C_2 | I_1) + P(M_2, C_3 | I_1)$$

1/3 of the time he randomizes & tells you nothing

$$= \overset{\frac{1}{3}}{P(C_1)} \overset{\frac{1}{2}}{P(M_2 | I_1, C_1)} +$$

2/3 of the time monty tells you where the car is

$$\overset{\frac{1}{3}}{P(C_2)} \overset{0}{P(M_2 | I_1, C_2)} +$$

$$\overset{\frac{1}{3}}{P(C_3)} \overset{1}{P(M_2 | I_1, C_3)}$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)(0) + \left(\frac{1}{3}\right) 1 = \frac{1}{2}$$

and $P(C_3 | I_1, M_2) = \dfrac{\left(\frac{1}{3}\right)\cdot(1)}{\left(\frac{1}{2}\right)} = \dfrac{2}{3}$ ✓