Discussion section, week of 11-15 May 2020

Transformations of random variables

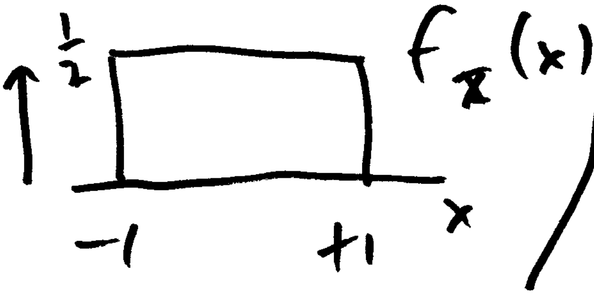
STAT 131
11 May 20
DD disc. section

①

You're working with a continuous random variable X with PDF $f_X(x)$, and you get interested in a new random variable Y that's related to X via $Y \doteq g(X)$ for some invertible function $g(\cdot)$ — Y is called a <u>transformed</u> version of X; how does the probability behavior of Y relate to that of X?

Example $X \sim$ Uniform$[-1, +1]$ (continuous)

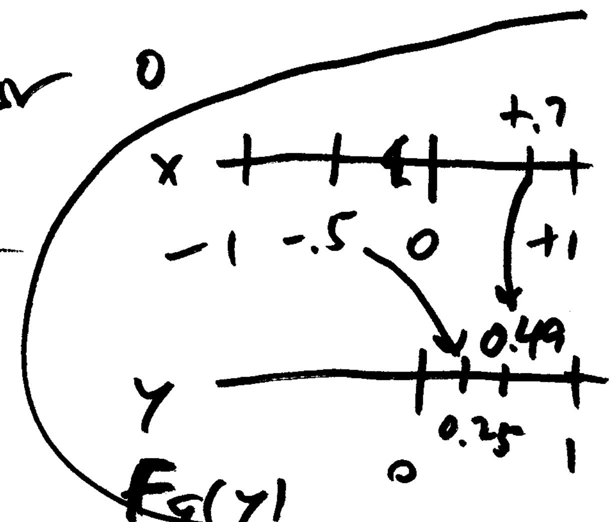
support of X is S_X



Set $Y = X^2$; PDF of Y?

First thing to note: $S'_Y = [0, 1]$ ← support of Y

Intuition: when you square an x between -1 and $+1$, the result moves closer to 0, so the PDF of Y should have lots of probability near 0



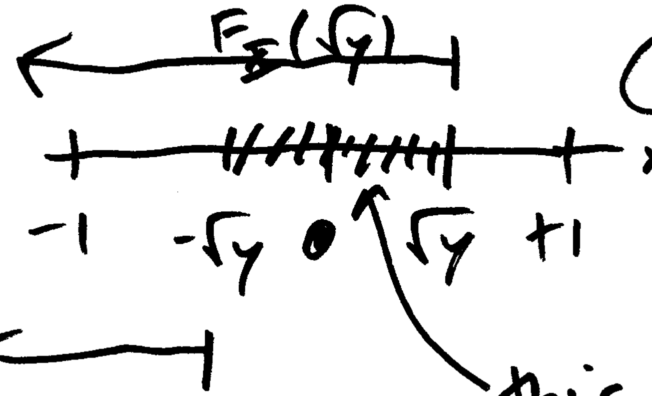
Method to get PDF of $Y = g(X)$, (1) first get CDF of Y in terms of CDF of X and then (2) differentiate $F_Y(y)$ in y to get PDF $f_Y(y)$

~~(for any $0 < y < 1$)~~

$$F_Y(y) = P(Y \le y) = P(\overbrace{X^2}^{g(X)} \le y)$$

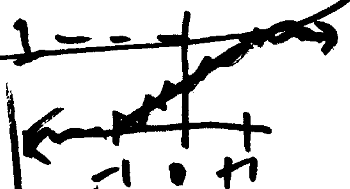
$= P(-\sqrt{y} \le X \le \sqrt{y})$ ⊛

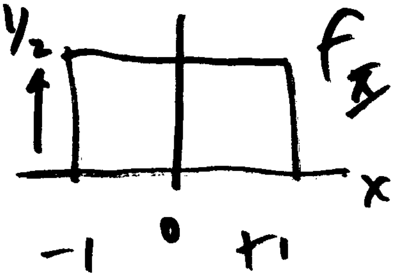
(inverting $g(X)$).



this is the probability ⊛ we want

$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$



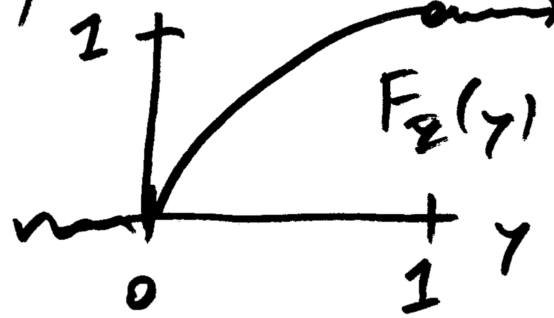


$$F_X(x) = \begin{cases} 0 & \text{for } x \le -1 \\ \int_{-1}^{x} \frac{1}{2}\, dt = \frac{x+1}{2} & \text{for } -1 \le x \le +1 \\ 1 & \text{for } x \ge +1 \end{cases}$$

for $(0 \le y \le 1)$

so $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$$= \left(\frac{1+\sqrt{y}}{2}\right) - \left(\frac{1-\sqrt{y}}{2}\right) = \sqrt{y}$$

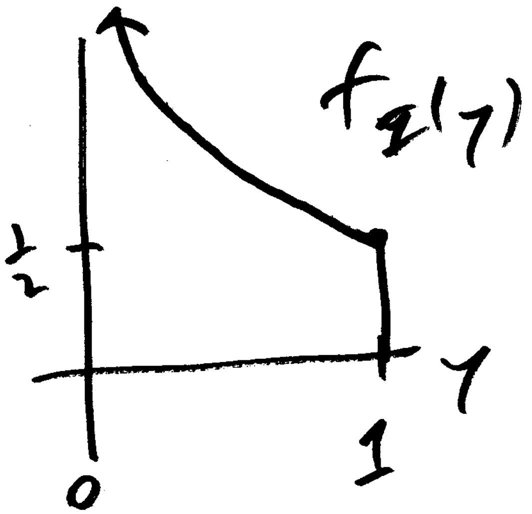


we could also have computed this as

$$P(-\sqrt{y} \le X \le +\sqrt{y}] = \int_{-\sqrt{y}}^{+\sqrt{y}} \underbrace{\frac{1}{2}}_{f_X(t)}\, dt = \sqrt{y} \checkmark$$

so for $0 \le y \le 1$, $F_Y(y) = \sqrt{y}$ and

therefore $f_Y(y) = \frac{d}{dy}\sqrt{y} = \frac{d}{dy} y^{\frac{1}{2}} = \frac{1}{2\sqrt{y}}$



Remarkably, the PDF of $Y = X^2$ is <u>unbounded</u> at 0: $\lim_{y \downarrow 0} \frac{1}{2\sqrt{y}} = +\infty$ (!)

(R demo)

And now, for an amazing use of this idea: general setup: X continuous r.v. with CDF $F_X(x)$ and PDF $f_X(x)$; $Y \doteq g(X)$ for invertible $g(\cdot)$

My favorite invertible function in this class is $g(x) = F_X(x)$; what happens with that choice of $g(\cdot)$?

$Y = F_X(X)$ (!)

~~note: $S_Y = (0,1)$~~

Let's just see what happens:

$$F_Y(y) = P(Y \le y) = P[F_X(X) \le y]$$

Now, since X is continuous, $F_X(\cdot)$, and strictly increasing, is invertible on the support S_X of X,

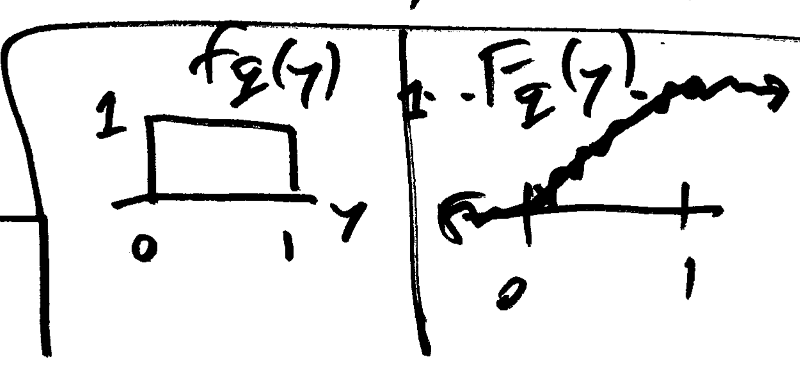
$$\text{so } F_Y(y) = P[F_X(X) \le y] = P[X \le F_X^{-1}(y)]$$

$$\overset{(!)}{=} F_X[F_X^{-1}(y)] \overset{(!)}{=} y$$

But there's only one random variable Y with CDF

$F_Y(y) = y$ (for $0 < y < 1$), namely

Uniform$(0,1)$ (!)



So what we have just shown is that if X has CDF $F_X(\cdot)$ then

$$Y = F_X(X) \sim \text{Uniform}(0,1) \; (!)$$

Now, to finish off the amazement by creating something extremely useful in data science:

let's run this argument backwards

$$Y = F_X(X)$$

$\uparrow$

Uniform(0, 1)

that makes me highly interested in the transformed random variable by hitting a Uniform(0,1)

$F_X^{-1}(Y)$; random variable with $F_X^{-1}(\cdot)$, do we recover X?

Let's see:

Let $U \sim$ Uniform$(0,1)$ and create ⑦

$W = F_X^{-1}(U)$; what's the CDF of W?

$$F_W(w) = P(W \le w) = P[F_X^{-1}(U) \le w]$$

But in this setup, because X is continuous, $F_X^{-1}(\cdot)$ is also invertible and strictly increasing. So we get

---

$$F_W(w) = P(F_X^{-1}(U) \le w) = P[U \le F_X(w)]$$

But $0 \le F_X(w) \le 1$, and we already worked out that a Uniform$(0,1)$ rv such as U has CDF $P(U \le u) = u$ (for $0 < u < 1$)

And finally, then, $F_W(w) = F_X(w)$;

two (continuous) rv with the same CDF must have the same PDF; so (the envelope, please)

X continuous r.v. with CDF $F_X(x)$ and PDF $f_X(x)$ $\longrightarrow$

$U \sim \text{Uniform}(0,1)$

$\downarrow$

$W = F_X^{-1}(U)$ has CDF $F_X(\cdot)$ and PDF $f_X(\cdot)$

$\boxed{\text{Q:}}$ Why is this useful?

$\boxed{\text{A:}}$ to <u>simulate</u> random draws (the foundation of the Monte Carlo method for learning about the world) from a PDF $f_X(x)$,

Algorithm (Inverse Probability Integral Transform) ⑨

(IPIT)

① Given PDF $f_X(x)$, compute CDF $F_X(x)$ and set $F_X(x_p) = p$

② compute $x_p = F_X^{-1}(p)$ by solving for x_p

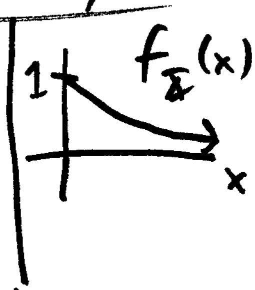
③ generate $U_1^*, \dots, U_M^* \overset{IID}{\sim} \text{Uniform}(0,1)$

④ Set $X_i^* = F_X^{-1}(U_i^*)$

Example (extra notes p. 81)

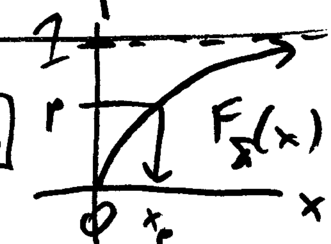
X = voltage in an electrical system

$$f_X(x) = \begin{cases} \dfrac{1}{(1+x)^2} & \text{for } x > 0 \\ 0 & \text{else} \end{cases}$$

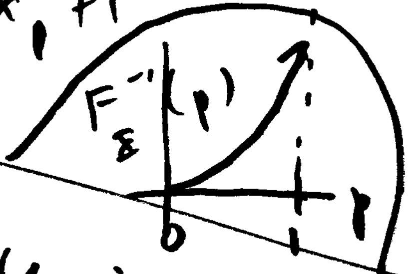


① for $x > 0$

$$F_X(x) = \int_0^x \frac{1}{(1+t)^2}\,dt = \frac{x}{x+1} \quad \boxed{W_a}$$



② $F_X(x_p) = p \rightarrow \frac{x_p}{x_p+1} = p \xrightarrow[x_p]{\text{solve for}} \boxed{x_p = \frac{p}{1-p}}$ (odds ratio (!))



③ Generate $U_i^* \overset{IID}{\sim}$ uniform$(0, 1)$ for $i = 1, \ldots, M$

(R demo)

④ Compute $X_i^* = \frac{U_i^*}{1 - U_i^*}$ $(i = 1, \ldots, M)$

(R demo)

In the early days of computing (mid to late 1940s for the theory (Turing, von Neumann, ...), early 1950s for the practice, pseudo-<u>uniform</u> "random" number generators were the first Monte Carlo algorithms created; IPIT greatly expanded the scope of simulation.