THT 2

4

$$\ell(\theta \mid \mathbf{y}) = \theta^{-n} I(\theta \geq m)$$

un-normalized PDF

likelihood

STAT 131

28 May 20

prior PDF

$$f_{\Theta}(\theta) = c\, \theta^{-(\alpha+1)} I(\theta \geq \beta)$$

Pareto(α, β)

$\begin{pmatrix} \alpha > 0 \\ \beta > 0 \end{pmatrix}$

DD office

1.5-hour session

(1)

$\ell(\theta \mid \mathbf{y})$ = un-normalized Pareto$(n-1, m)$

m = max$(\mathbf{y})$

$\mathbf{y} = (y_1, \dots, y_n)$

(f'n of θ for fixed $\mathbf{y}$)

(once $\mathbf{y}$ observed)

$$\boxed{f_{(\Theta \mid \mathbf{y})}(\theta \mid \mathbf{y})} = c \cdot f_{\Theta}(\theta) \cdot \boxed{p(\mathbf{y} \mid \theta)}$$

$p(\mathbf{y} \mid \theta) \to \ell(\theta \mid \mathbf{y})$

$$f_{(\Theta \mid \mathbf{y})}(\theta \mid \mathbf{y}) = c \cdot f_{\Theta}(\theta) \cdot \boxed{\ell(\theta \mid \mathbf{y})}$$

posterior PDF = (constant) $\cdot$ (Pareto prior PDF) (Pareto likelihood PDF)

$$f_{(\Theta|z)}(\theta|z) = c \cdot \theta^{-(\alpha+1)} I(\theta \geq \beta) \cdot \theta^{-n} I(\theta \geq m)$$

$$= c\, \theta^{-[(\alpha+n)+1]} I(\theta \geq \beta \text{ and } \theta \geq m)$$

$$I(C) \cdot I(D) = I(C \text{ and } D)$$

$$= c\, \theta^{-[(\alpha+n)+1]} \cdot I[\theta \geq \max(\beta, m)]$$

$$\text{Pareto}(\theta | A, B) \downarrow c\, \theta^{-(A+1)} I(\theta \geq B)$$

$$\begin{pmatrix} A > 0 \\ B > 0 \end{pmatrix}$$

$$= \text{Pareto}[\alpha + n, \max(\beta, m)]$$

= posterior PDF for θ given z

$$= f_{(\Theta|z)}(\theta|z)$$

$\boxed{\text{likelihood as f'n of } \theta \text{ for fixed } y}$ ① = un-normalized Pareto PDF

② if we choose prior PDF to also be a Pareto, something wonderful happens:

③ the product of two Paretos is another Pareto; this makes the Pareto prior conjugate to the Uniform$(0, \theta)$ likelihood (Bayes 1760)

$$(\theta \mid \alpha, \beta) \sim \text{Pareto}(\alpha, \beta)$$

$$(Y_i \mid \theta) \overset{\text{IID}}{\sim} \text{Uniform}(\theta) \quad (i = 1, \dots, n) \qquad y = (y_1, \dots, y_n)$$

$$m = \max(y)$$

$$(\theta \mid y) \sim \text{Pareto}[\alpha + n, \max(\beta, m)]$$

name!
Howard Raiffa
Art Schaiffer
(late 1950s)

$$x = \begin{bmatrix} 2.8 \\ \vdots \\ 1 \\ 0.4 \end{bmatrix}$$

max $m = 5.1$

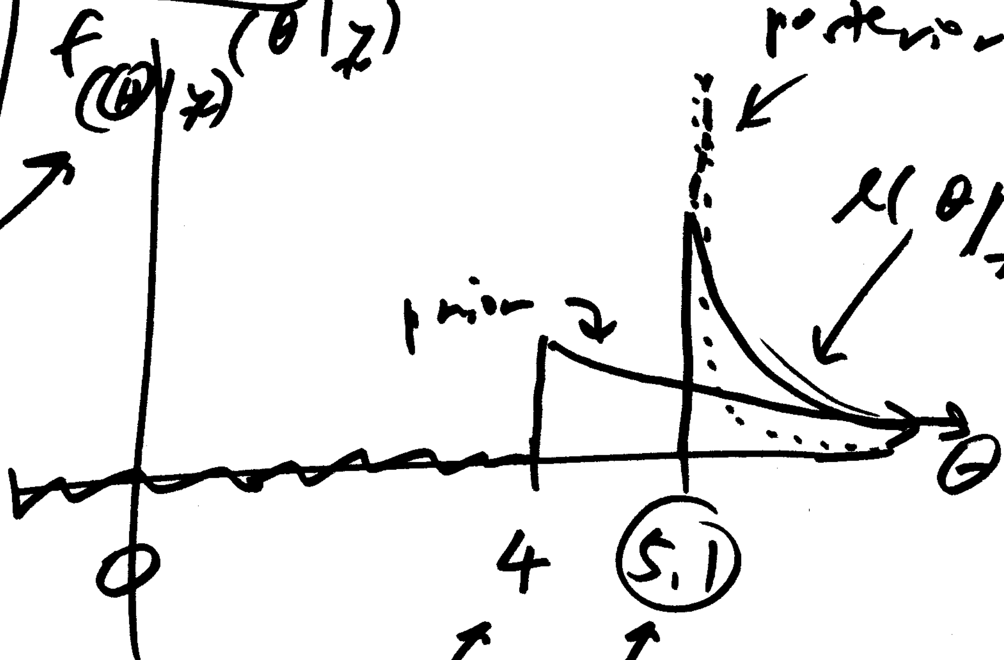
$\alpha = 2.5$

$\beta = 4$

$(\theta > 0)$

$f(\theta \mid x)$

Pareto$[\underline{\alpha + n}, \boxed{\max(\overset{4}{\beta}, \overset{5.1}{m})}]$



β → 4

$\max(\beta, m)$ → 5.1

biggest SD prior	sample size data	smallest SD posterior
2.5	11	13.5
α	n	$\underline{\underline{\alpha + n}}$

$=$

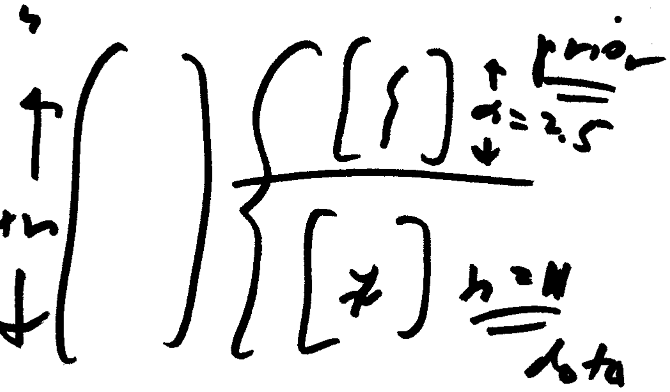
data set "worth n votes"

posterior sample size

(spread of PDF for θ) ↔ uncertainty about θ

prior "worth α votes"

prior sample size



$$f_{(\theta|\tilde{y})}(\theta \mid \tilde{y}) = \text{Pareto}\Big[\underbrace{\alpha + n}_{13.5},\ \underbrace{\max(\beta, y_{(n)})}_{5.1}\Big]$$

$\alpha = 2.5$, $n = 11$, $\beta = 4$, $y_{(n)} = 5.1$

$$\theta \sim \text{Pareto}(A, B) \rightarrow E(\theta) = \frac{AB}{A-1} \quad A > 1$$

$$V(\theta) = \frac{AB^2}{(A-1)^2(A-2)} \quad A > 2$$

posterior mean $= \dfrac{AB}{A-1} = \cdots = \dfrac{(13.5)(5.1)}{12.5}$

$A = 13.5$

$B = 5.1$

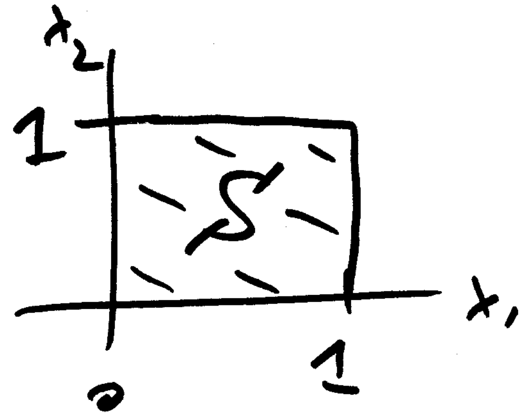
$$= B \cdot \left(\frac{A}{A-1}\right) = (5.1)\left(\frac{13.5}{12.5}\right) = 5.5$$

posterior SD $= \sqrt{\dfrac{AB^2}{(A-1)^2(A-2)}} \doteq 0.442$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = 4x_1x_2 \boxed{I(0<x_1<1 \overset{\text{and}}{,} 0<x_2<1)}$$

$\underset{\sim}{X} \to (X_1, X_2)$

$\boxed{2(c)}$



$\underset{\sim}{Y} = (Y_1, Y_2)$

$Y_1 = X_1$

$Y_2 = X_1 \cdot X_2$ $\longleftrightarrow$ $y_1 = x_1$, $y_2 = x_1 \cdot x_2$

$x_2 = \dfrac{y_2}{x_1} = \dfrac{y_2}{y_1}$

$x_1 = y_1$

$x_2 = \dfrac{y_2}{y_1}$

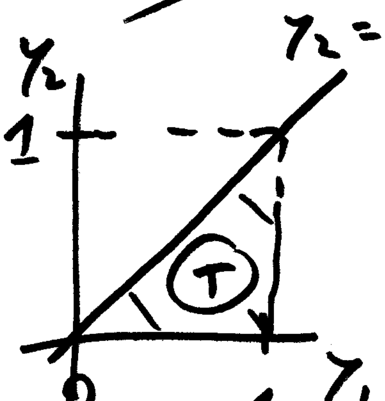
$0 < x_1 < 1$

$0 < y_1 < 1$

$0 < x_2 < 1$

$0 < \dfrac{y_2}{y_1} < 1$

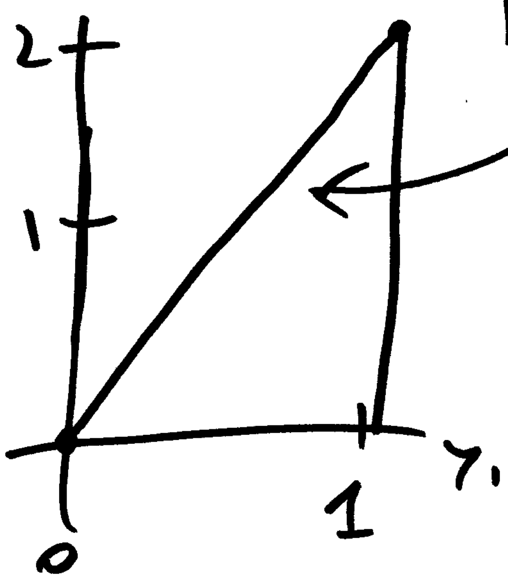
$0 < y_2 < y_1$



$T = \{(y_1, y_2):$ $0 < y_1 < 1,$ $0 < y_2 < y_1\}$

$(\boxed{0 < y_2 < y_1} < 1)$

$$f_{Y_1}(y_1) = \begin{cases} 2y_1 & \text{for } 0<y_1<1 \\ 0 & \text{else} \end{cases}$$



marginal PDF for Y_1

$$f_{Y_2}(y_2) = \begin{cases} -4\,y_2 \log y_2 & \text{for } 0<y_2<1 \\ 0 & \text{else} \end{cases}$$

$$\lim_{y_2 \downarrow 0} y_2 \log y_2 = \lim_{y_2 \downarrow 0} \frac{y_2}{\frac{1}{\log y_2}} = 0$$