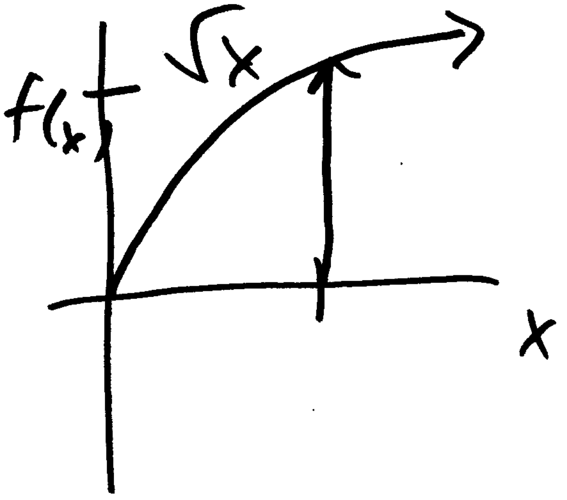
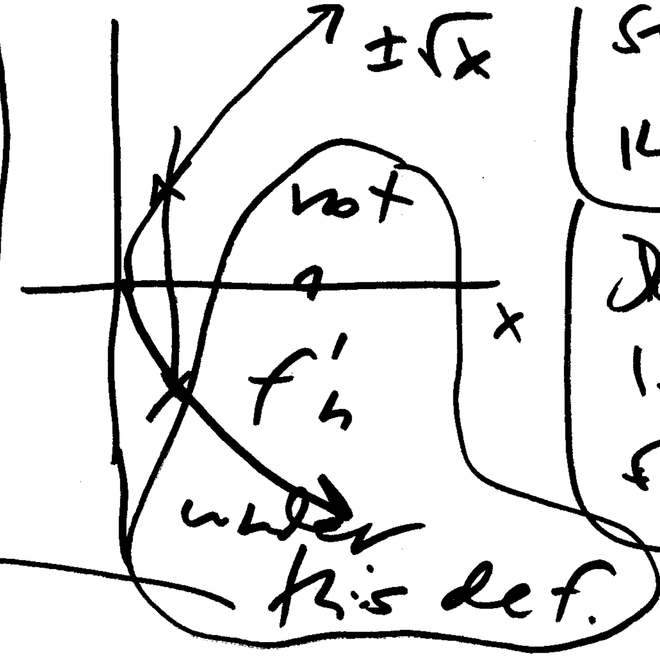
STAT 131
14 May 20

DD office
1.5-hour
session





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one def:

A necessary condition for $f : A \to B$ to be a function is that for all $x \in A$ $f(x) \in B$ is unique

Quiz 6

$$F_Y^{-1}(p) = \frac{1}{2}\left(\sqrt{8p+1} - 1\right)$$

$0 \le p \le 1$

Y has CDF $F_Y(y)$ and PDF $f_Y(y)$

IPIT:

1. draw $U_1^*, \dots, U_n^* \stackrel{IID}{\sim}$ Uniform$(0,1)$

$0 \le U_i^* \le 1$

2. compute

IID PDF $f_Y(y)$ $\longrightarrow$ $Y_i^* = \frac{1}{2}\left(\sqrt{8U_i^*+1} - 1\right)$

X r.v. with PDF $f_X(x)$ and CDF $F_X(x)$
continuous

$U_1^*, \dots, U_n^* \overset{IID}{\sim}$ Uniform$(0, 1)$

IPFT

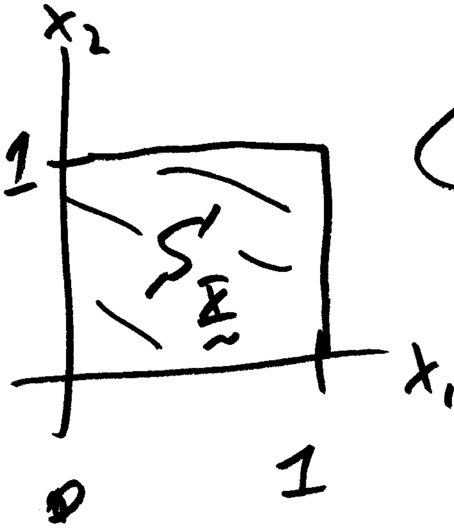
$F_X^{-1}(U_1^*), \dots, F_X^{-1}(U_n^*) \overset{IID}{\sim}$ with PDF $f_X(x)$ and CDF $F_X(x)$

THT 2 # 2(a), (c)

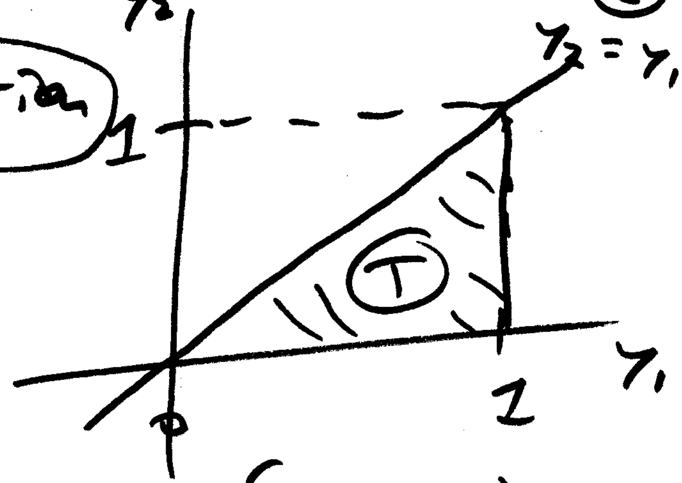
$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \begin{cases} 4x_1x_2 & \text{for } \left\{\begin{matrix} 0 < x_1 < 1 \\ 0 < x_2 < 1 \end{matrix}\right\} = S_X \\ 0 & \text{else} \end{cases}$$

$\underset{\sim}{X} = (X_1, X_2)$

$\underset{\sim}{Y} = (Y_1, Y_2)$; $Y_1 = X_1$, $Y_2 = X_1 \cdot X_2$



transformation $\longrightarrow$



$T = \{(y_1, y_2) : 0 < y_2 < y_1 < 1\}$

$y_1 = h_1(x_1, x_2) = x_1$

$y_2 = h_2(x_1, x_2) = x_1 \cdot x_2 \longrightarrow x_2 = \dfrac{y_2}{x_1} = \dfrac{y_2}{y_1}$

$x_1 = h_1^{-1}(y_1, y_2) = y_1$

$x_2 = h_2^{-1}(y_1, y_2) = \dfrac{y_2}{y_1}$

$0 < x_1 < 1 \longrightarrow 0 < y_1 < 1$

$0 < x_2 < 1 \longrightarrow 0 < \dfrac{y_2}{y_1} < 1$

$0 < \dfrac{y_2}{y_1}$ $\quad$ $T = S_{\underset{\sim}{X}}$ $\quad$ $\dfrac{y_2}{y_1} < 1$ $\quad$ $y_2 < y_1$

$\longrightarrow 0 < y_2 < y_1 < 1$

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} 4 x_1 x_2 & 0 < x_1 < 1 \\ & 0 < x_2 < 1 \\ 0 & \text{else} \end{cases}$$

2(a) Are X_1, X_2 independent in this joint PDF?

① (necessary for independence)

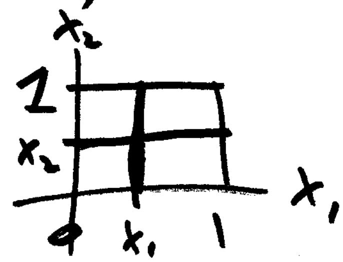
$S_{\underset{\sim}{X}}$ is the rectangle $= \begin{pmatrix} \text{unit} \\ \text{square} \end{pmatrix}$

② $\begin{pmatrix} \text{also} \\ \text{need} \end{pmatrix}$ $f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$

for $0 < x_1 < 1$

$$f_{X_1}(x_1) = \int_{S_{\underset{\sim}{X}}} f_{X_1, X_2}(x_1, x_2)\, dx_2$$

$$= \int_0^1 4 x_1 x_2 \, dx_2 = 2 x_1$$



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So $f_{X_1}(x_1) = \begin{cases} 2x_1 & \text{for } 0 < x_1 < 1 \\ 0 & \text{else} \end{cases}$

$f_{X_2}(x_2)$ = same as $f_{X_1}(x_1)$

but with x_2 in place of x_1

check: for $0 < x_2 < 1$

$$f_{X_2}(x_2) = \int_0^1 (4x_1x_2)\,dx_1$$

$$\underbrace{f_{X_1,X_2}(x_1, x_2)} = f_{X_1}(x) \cdot f_{X_2}(x_2)$$

$4x_1x_2$ = ✓ $2x_1$ | $2x_2$

for $0 < x_1 < 1$, $0 < x_2 < 1$ | for $0 < x_1 < 1$ | for $0 < x_2 < 1$

(X_1, X_2) continuous joint PDF $f_{\underset{\sim}{X}}(\underset{\sim}{x})$, and support $S_{\underset{\sim}{X}}$ ⑥

$(n=2)$

$Y_1 = h_1(X_1, X_2) = X_1$

$Y_2 = h_2(X_1, X_2) = X_1 \cdot X_2$

inverse transform

$x_1 = h_1^{-1}(y_1, y_2) = y_1$

$x_2 = h_2^{-1}(y_1, y_2) = \frac{y_2}{y_1}$

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = f_{Y_1, Y_2}(y_1, y_2)$$

$$= \begin{cases} f_{\underset{\sim}{X}}\left[h_1^{-1}(\underset{\sim}{y}), h_2^{-1}(\underset{\sim}{y})\right] |J| & \text{for } \underset{\sim}{y} \in T \\ 0 & \text{else} \end{cases}$$

$$J = \det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial y_1} & \frac{\partial h_1^{-1}}{\partial y_2} \\ \frac{\partial h_2^{-1}}{\partial y_1} & \frac{\partial h_2^{-1}}{\partial y_2} \end{bmatrix}$$

$h_1^{-1}(y_1, y_2) = y_1$

$h_2^{-1}(y_1, y_2) = \frac{y_2}{y_1}$

$= y_2 \cdot y_1^{-1}$

$$\frac{\partial}{\partial y_1} h_1^{-1} = 1 \qquad \frac{\partial h_1^{-1}}{\partial y_2} = 0 \qquad \frac{\partial}{\partial y_1} h_2^{-1} = -\frac{y_2}{y_1^2}$$

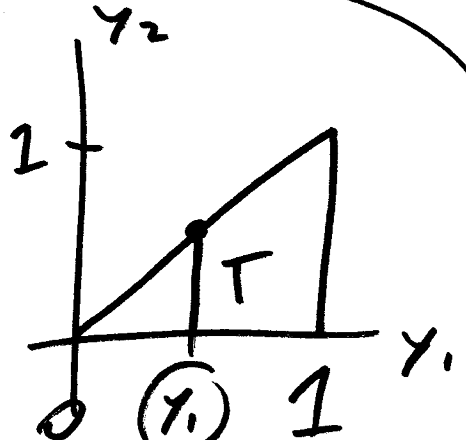
$$\frac{\partial}{\partial y_2} h_2^{-1} = \frac{1}{y_1} \qquad |J| = \left|\det \begin{bmatrix} 1 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{bmatrix}\right| = \left|\frac{1}{y_1}\right| = \frac{1}{y_1}$$

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \begin{cases} f_{\underset{\sim}{X}}\left[h_1^{-1}(\underset{\sim}{y}), h_2^{-1}(\underset{\sim}{y})\right]|J| & \text{for } \underset{\sim}{y} \in \mathcal{T} \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} 4 \cdot y_1 \cdot \frac{y_2}{y_1} \cdot \frac{1}{y_1} & \text{for } 0 < y_2 < y_1 < 1 \\ = 4\frac{y_2}{y_1} & \\ 0 & \text{else} \end{cases}$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) \overset{(x_1, x_2)}{} = 4x_1x_2 \text{ for } \underset{\sim}{x} \in \mathcal{S}_{\underset{\sim}{X}}$$

$$\iint_T f_{\underset{\sim}{Y}}(y)\,dy = \iint_T 4\frac{y_2}{y_1}\,dy_2\,dy_1 =$$



$$= \int_0^1\left[\int_0^{y_1} 4\frac{y_2}{y_1}\,dy_2\right]dy_1$$

$$= \int_0^1 \frac{4}{y_1}\left[\int_0^{y_1} y_2\,dy_2\right]dy_1$$

$$= \int_0^1 \frac{4}{y_1}\left(\frac{y_2^2}{2}\Big|_0^{y_1}\right)dy_1$$

$f_{Y_1}(y_1) = ?$

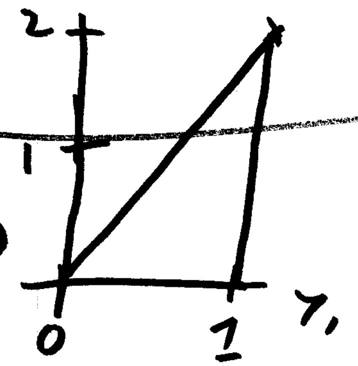
$$= \int_0^1 \left(\frac{4}{y_1}\right)\left(\frac{y_1^2}{2}\right)dy_1$$

$$= \begin{cases} (*) & \text{for } 0<y_1<1 \\ 0 & \text{else} \end{cases}$$

$$= \int_0^1 \underline{2y_1}\,dy_1 = 1 \checkmark$$

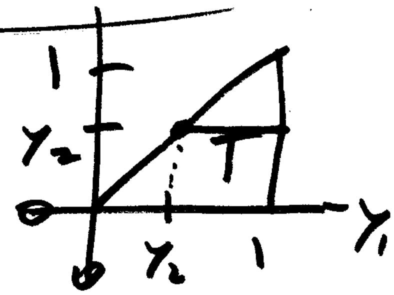
for $0<y_1<1$

$$(*) = \int_0^{y_1} 4\frac{y_2}{y_1}\,dy_2 = 2y_1 \longrightarrow$$



✓

$$f_{Y_2}(y_2) = \begin{cases} (**) & \text{for } 0 < y_2 < 1 \\ 0 & \text{else} \end{cases}$$



$\underline{\text{for } 0 < y_2 < 1}$

$$f_{Y_2}(y_2) = \int_{y_2}^{1} 4\frac{y_2}{y_1}\,dy_1$$

$$= 4y_2 \int_{y_2}^{1} \frac{1}{y_1}\,dy_1$$

$$= 4y_2 \log y_1 \Big|_{y_2}^{1}$$

$$= 4y_2 \left[0 - \log y_2 \right]$$

$$= -4y_2 \log y_2 \quad ✓$$