$\boxed{7}$ Double integration | This may also be

new to you, and it requires two new skills: (1) Sweeping over one variable while holding the other variable constant, and (2) visualizing the support set S of the function you're integrating. Example

$$f(x,y) = \begin{cases} c(x^2+y) & \text{for } \overbrace{0 \le y \le 1-x^2}^{S} \\ 0 & \text{else} \end{cases}$$

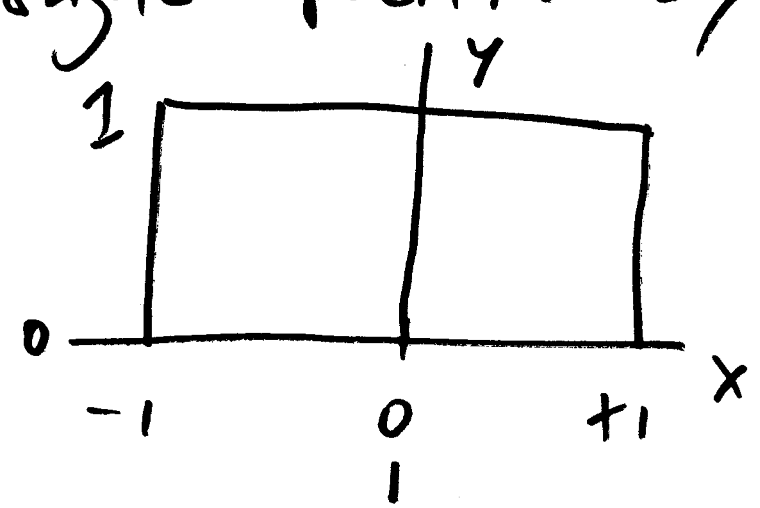
(for some constant $c > 0$)

(2) (above) first | Q: Where in the (x,y) plane are the points satisfying $0 \le y \le 1 - x^2$?

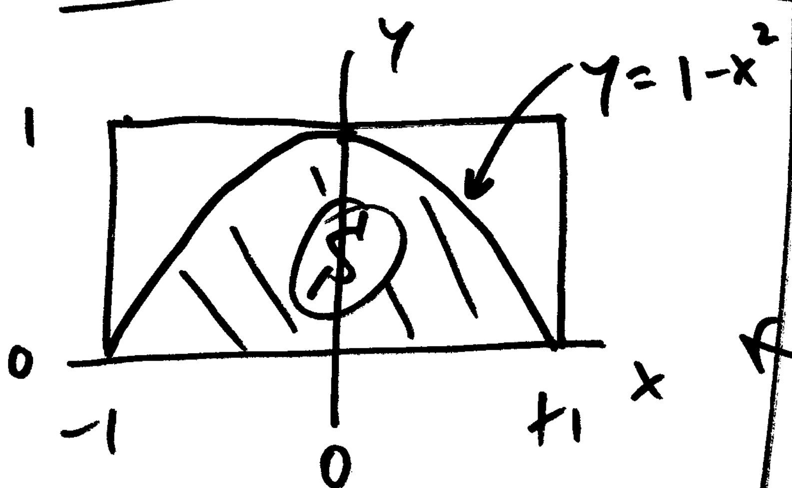
A: Step 1 First work out what's called the marginal support in the x and y directions separately.

Looking at the inequalities $0 \le y \le 1 - x^2$ only from the point of view of x yields $0 \le 1 - x^2$, from which $x^2 \le 1$, which is true for all $-1 \le x \le +1$, so $[-1, +1]$ is the marginal support for x. Now as x goes from -1 to $+1$, $(1 - x^2)$ goes from 0 to $+1$ and back down again to 0, so the marginal support for y is $0 \le y \le 1$ $[0, 1]$.

Step 2 / Plot the rectangle specified by the two marginal support sets:



Step 3 Not all the points in the rectangle in Step 2 are in what's called the bivariate support set S, just the points where $0 \le y \le 1 - x^2$; it's clear that we should add the curve $y = 1 - x^2$ to the rectangle:



But this is just a bowl-shaped-down parabola.

Step 4

Now, finally, (y between 0 and $(1-x^2)$) and inside the rectangle is the points below the parabola, and we have our bivariate support set S.

Wd RegionPlot[y < 1 - x^2, {x, -1, 1}, {y, 0, 1}]

Now let's compute the double integral

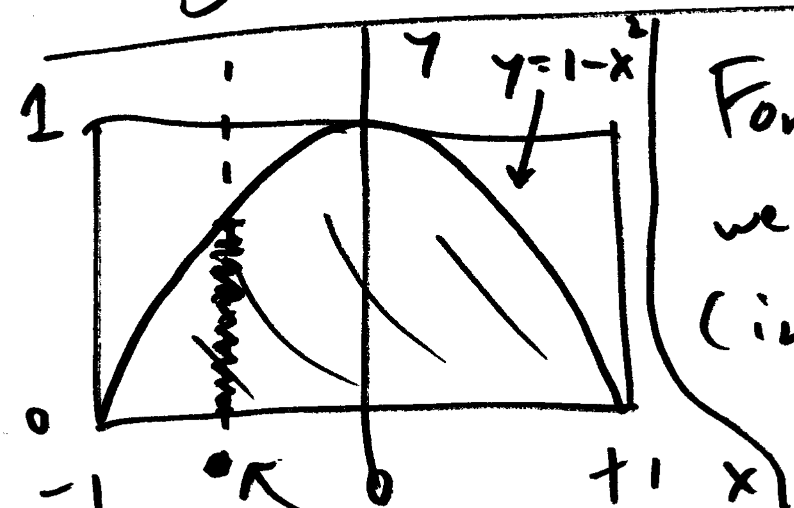
$$\iint_S f(x,y)\,dx\,dy.$$

The first thing to say is that, just as with mixed partial derivatives, you can integrate over x and y in either order and you will always get the same thing: $\iint_S f(x,y)\,dx\,dy = \iint_S f(x,y)\,dy\,dx$

One order is often easier to work with than the other; you'll get good at choosing the easier order (if any) with practice; but you can check your work by doing it both ways (if the answers differ, you've done something wrong).

let's first compute $\iint_S c(x^2+y)\,dy\,dx$.

With x as the <u>outer</u> variable of integration, the idea is to write the double integral as $\int_{-1}^{+1}\left[\int_{\boxed{?}_1}^{\boxed{?}_2} c(x^2+y)\,dy\right]dx$, using the marginal support for x $[-1,+1]$ to define the limits of integration for the <u>outer</u> integral; now we just need to figure out the inner limits of integration $\boxed{?}_1$, $\boxed{?}_2$.



pick an x between -1 and $+1$

For any given $-1 \le x \le +1$, we don't want to "sweep" (integrate) over y from 0 to 1 (its marginal support);

we only want to sweep over y values in the <u>bivariate</u> support set, which (for the chosen x) is y between 0 and $(1-x^2)$.

Thus our integral is $\circledast = \int_{-1}^{+1} \left[\int_0^{1-x^2} c(x^2+y)\,dy \right] dx$.

The way you compute $\circledast$ is to (a) evaluate the inner integral first, and then (b) do the outer integral using the result in (a):

(a) $\int_0^{1-x^2} c(x^2+y)\,dy = c\left(x^2 y + \frac{y^2}{2} \right)_{y=0}^{y=1-x^2}$

$= c\left\{ \left[x^2(1-x^2) + \frac{(1-x^2)^2}{2} \right] - (0+0) \right\}$

$= \frac{c}{2}(1-x^4)$ after a bit of simplification

(b) $\circledast = \int_{-1}^{+1} \frac{c}{2}(1-x^4)\,dx = \frac{c}{2}\left(x - \frac{x^5}{5}\right)\Bigg|_{x=-1}^{x=+1}$

$= \frac{4c}{5}$ after another bit of simplification.

Let's check our work by also computing

$$\iint_{S} f(x,y)\,\underset{\substack{\uparrow \\ \text{(reverse order of integration)}}}{\underline{dx\,dy}} = \iint_{S} c(x^2+y)\,dx\,dy$$

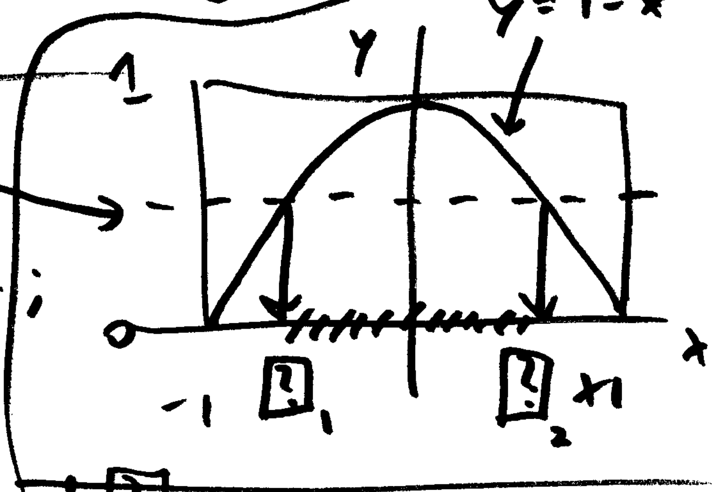
---

we can use the same steps (a) and (b) above:

$$(**) = \int_0^1 \left[\int_{\boxed{?}_1}^{\boxed{?}_2} c(x^2+y)\,dx \right] dy$$

marginal limits on outer integral

pick a y between 0 and 1; we only want to sweep over x between $\boxed{?}_1$ and $\boxed{?}_2$



32

To identify $\boxed{?}_1$ and $\boxed{?}_2$ we need to solve $(y = 1 - x^2)$ backwards for x (this is <u>inverting</u> the function $g(x) = 1 - x^2$):

$$y = 1 - x^2 \rightarrow x^2 = 1 - y \rightarrow x = \pm\sqrt{1-y}$$

so (**) $= \displaystyle\int_0^1 \left[\int_{-\sqrt{1-y}}^{+\sqrt{1-y}} c(x^2 + y)\, dx \right] dy.$

The inner integral has antiderivative (in x) $c\left(\frac{x^3}{3} + xy\right)$, so it becomes

$$\int_{-\sqrt{1-y}}^{+\sqrt{1-y}} c(x^2+y)\, dx = c\left(\frac{x^3}{3} + xy\right)\Bigg|_{x=-\sqrt{1-y}}^{x=+\sqrt{1-y}}$$

$= \frac{2c}{3}(2y+1)\sqrt{1-y}$ (considerably less pleasant than $dy\,dx$)
(after simplifying)

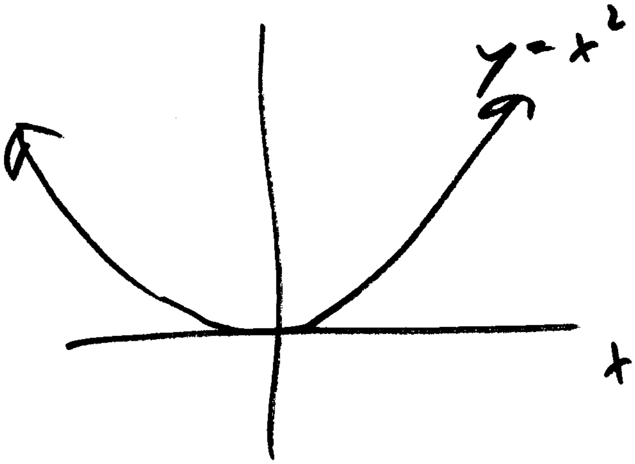
and now finally $(**) = \int_0^1 \frac{2c}{3}(2y+1)\sqrt{1-y}\,dy = \frac{4c}{5}$

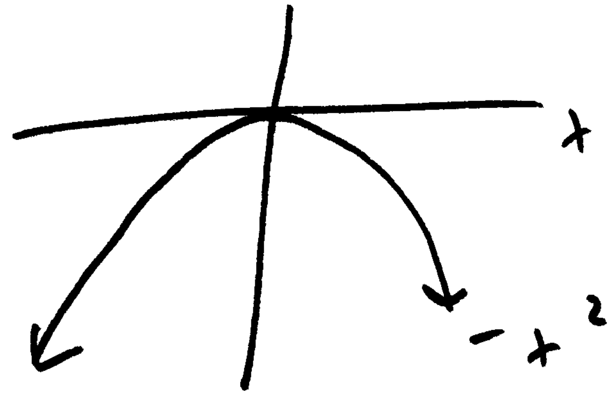
as before.

Some helpful $\boxed{W\alpha}$ code here:

```
integrate integrate c(x^2+y) for y
  from 0 to (1-x^2) for x from -1 to 1
```

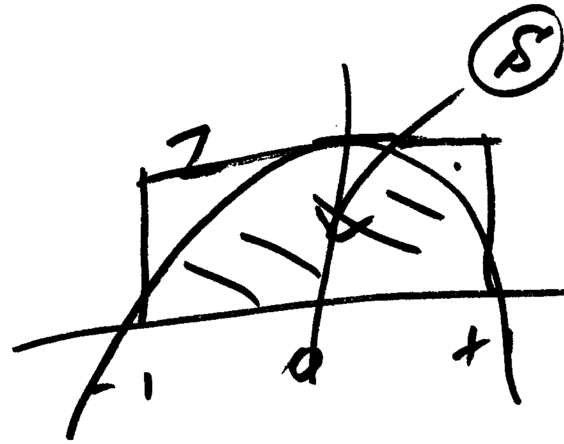
```
integrate integrate c(x^2+y) for x
  from -sqrt(1-y) to sqrt(1-y)
    for y from 0 to 1
```





Scratch work: disc sec. week 5 (JJ) (27 Apr 20)

$y \le 1-x^2$



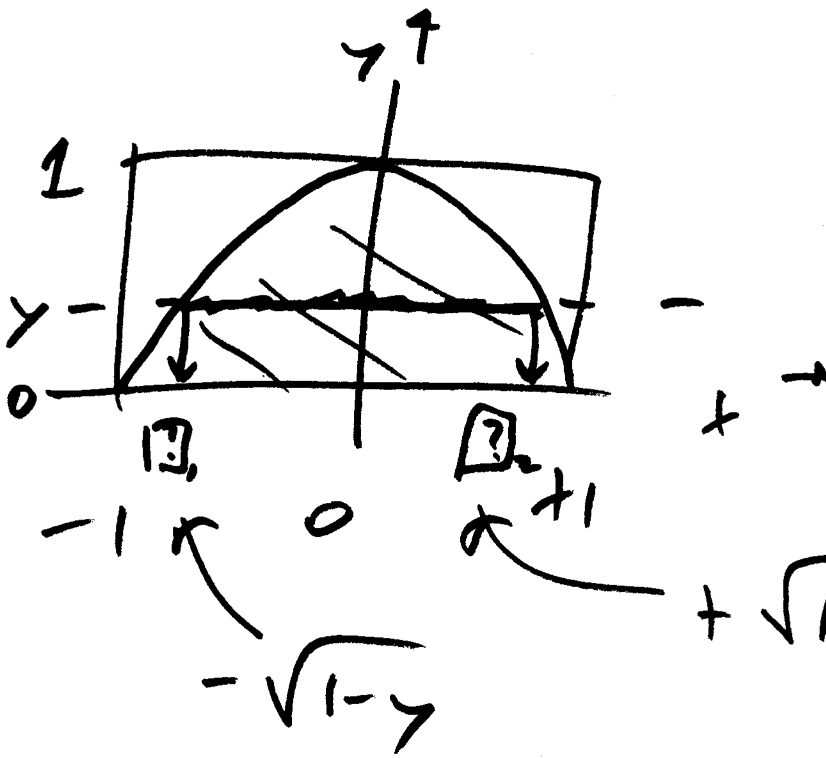
$$\int_0^{1-x^2} c(x^2+y)\,dy \neq cx^2+cy$$

$$= c\left(\cancel{c}x^2y + \cancel{c}\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1-x^2}$$

$$= c\left[\left(x^2(1-x^2)+\frac{(1-x^2)^2}{2}\right) - (0+0)\right]$$

②

$$\int_0^1 \left[\int_{\cancel{\boxed{?}_1}\, -\sqrt{1-y}}^{\cancel{\boxed{?}_2}\, +\sqrt{1-y}} c(x^2+y)\, dx \right] dy$$



$y = 1 - x^2 \rightarrow$

$x^2 = 1 - y$

$x = \pm\sqrt{1-y}$

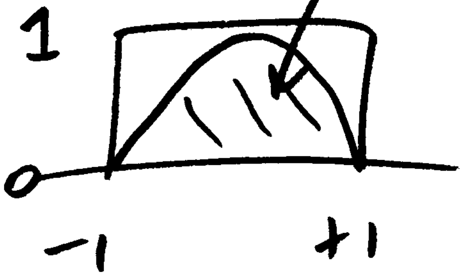
$f_X(x)$ is a PDF for cont. r.v. X iff

① $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

and ② $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1$$

$$\underline{\underline{\iint_S c\,(x^2+y)}}\,dy\,dx = 1 = \frac{4c}{5}$$



$$c = \frac{5}{4}$$

$$\frac{5}{4}(x^2+y) \text{ on } S$$

$f_{X,Y}(x,y) =$ is a

bivariate PDF